AR(2) model

h-steps ahead forecasting distributions

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The AR(2) Model with Intercept

$$y_t = \mu + \phi_1 \, y_{t-1} + \phi_2 \, y_{t-2} + \varepsilon_t, \qquad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$$

Goal: By repeatedly substituting the AR(2) equation into itself, express y_t as

$$y_t = C_k \cdot \mu + A_k \, y_{t-k} + B_k \, y_{t-k-1} + \sum_{j=0}^{k-1} A_j \, \varepsilon_{t-j}$$

where the coefficients satisfy the **recursion**:

$$\boxed{A_{k+1} = \phi_1 A_k + B_k, \qquad B_{k+1} = \phi_2 A_k, \qquad C_{k+1} = C_k + A_k}$$

with initial values $A_1 = \phi_1$, $B_1 = \phi_2$, $C_1 = 1$ (and $A_0 = 1$).

Note on ε coefficients

The coefficient of ε_{t-j} in the expansion is A_j, since each substitution of y_{t-k} introduces $A_k \, \varepsilon_{t-k}$.

Coefficient Table

Step k	A_k (coeff. of y_{t-k})	B_k (coeff. of y_{t-k-1})	C_k (coeff. of μ)
1	ϕ_1	ϕ_2	1
2	$\phi_1^2 + \phi_2$	$\phi_1\phi_2$	$1 + \phi_1$
3	$\phi_1^3 + 2\phi_1\phi_2$	$\phi_1^2\phi_2 + \phi_2^2$	$1 + \phi_1 + \phi_1^2 + \phi_2$
4	$\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2$	$\phi_1^3\phi_2 + 2\phi_1\phi_2^2$	$C_3 + A_3$
5	$\phi_1^5 + 4\phi_1^3\phi_2 + 3\phi_1\phi_2^2$	$\phi_1^4\phi_2 + 3\phi_1^2\phi_2^2 + \phi_2^3$	$C_4 + A_4$
6	$\phi_1^6 + 5\phi_1^4\phi_2 + 6\phi_1^2\phi_2^2 + \phi_2^3$	$\phi_1^5\phi_2 + 4\phi_1^3\phi_2^2 + 3\phi_1\phi_2^3$	$C_5 + A_5$

Note: the coefficients of A_k follow a Fibonacci-like pattern driven by (ϕ_1, ϕ_2).

Step 1 $\rightarrow$ Step 2: y_t in terms of y_{t-2} and y_{t-3}

Starting from

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$$

substitute $y_{t-1} = \mu + \phi_1 y_{t-2} + \phi_2 y_{t-3} + \varepsilon_{t-1}$:

$$\begin{aligned} y_t &= \mu + \phi_1\big(\mu + \phi_1 y_{t-2} + \phi_2 y_{t-3} + \varepsilon_{t-1}\big) + \phi_2 y_{t-2} + \varepsilon_t \\ &= \mu(1+\phi_1) + (\phi_1^2 + \phi_2)\, y_{t-2} + \phi_1\phi_2\, y_{t-3} + \varepsilon_t + \phi_1 \varepsilon_{t-1} \end{aligned}$$

Verification of recursion

$A_2 = \phi_1 A_1 + B_1 = \phi_1^2 + \phi_2$ ✓ $B_2 = \phi_2 A_1 = \phi_1\phi_2$ ✓ $C_2 = 1 + \phi_1$ ✓

Step 2 $\rightarrow$ Step 3: y_t in terms of y_{t-3} and y_{t-4}

From the previous step, substitute $y_{t-2} = \mu + \phi_1 y_{t-3} + \phi_2 y_{t-4} + \varepsilon_{t-2}$:

$$
\begin{aligned}
y_t &= \mu(1+\phi_1) + (\phi_1^2+\phi_2)\big(\mu + \phi_1 y_{t-3} + \phi_2 y_{t-4} + \varepsilon_{t-2}\big) \\
&\quad + \phi_1\phi_2\, y_{t-3} + \varepsilon_t + \phi_1\varepsilon_{t-1} \\
&= \mu\big(1+\phi_1+\phi_1^2+\phi_2\big) + \big(\phi_1^3 + 2\phi_1\phi_2\big)\, y_{t-3} \\
&\quad + \big(\phi_1^2\phi_2 + \phi_2^2\big)\, y_{t-4} + \varepsilon_t + \phi_1\varepsilon_{t-1} + (\phi_1^2+\phi_2)\varepsilon_{t-2}
\end{aligned}
$$

Verification

$A_3 = \phi_1(\phi_1^2+\phi_2) + \phi_1\phi_2 = \phi_1^3 + 2\phi_1\phi_2$ ✓ $B_3 = \phi_2(\phi_1^2+\phi_2) = \phi_1^2\phi_2 + \phi_2^2$ ✓

Step 3 → Step 4: y_t in terms of y_{t-4} and y_{t-5}

Substitute $y_{t-3} = \mu + \phi_1 y_{t-4} + \phi_2 y_{t-5} + \varepsilon_{t-3}$:

$$
\begin{aligned}
y_t &= \mu\left(1 + \phi_1 + \phi_1^2 + \phi_2\right) + \left(\phi_1^3 + 2\phi_1\phi_2\right)\left(\mu + \phi_1 y_{t-4} + \phi_2 y_{t-5} + \varepsilon_{t-3}\right) \\
&\quad + \left(\phi_1^2\phi_2 + \phi_2^2\right) y_{t-4} + \varepsilon_t + \phi_1\varepsilon_{t-1} + (\phi_1^2 + \phi_2)\varepsilon_{t-2} \\
&= \mu\, C_4 + \left(\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2\right) y_{t-4} \\
&\quad + \left(\phi_1^3\phi_2 + 2\phi_1\phi_2^2\right) y_{t-5} + \varepsilon_t + \phi_1\varepsilon_{t-1} + (\phi_1^2 + \phi_2)\varepsilon_{t-2} + (\phi_1^3 + 2\phi_1\phi_2)\varepsilon_{t-3}
\end{aligned}
$$

where $C_4 = 1 + \phi_1 + \phi_1^2 + \phi_2 + \phi_1^3 + 2\phi_1\phi_2$.

Verification

$A_4 = \phi_1(\phi_1^3 + 2\phi_1\phi_2) + (\phi_1^2\phi_2 + \phi_2^2) = \phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2 \;\checkmark$

Step 4 → Step 5: y_t in terms of y_{t-5} and y_{t-6}

Substitute $y_{t-4} = \mu + \phi_1 y_{t-5} + \phi_2 y_{t-6} + \varepsilon_{t-4}$:

$$\begin{aligned} y_t = {} & \mu\, C_5 + A_5\, y_{t-5} + B_5\, y_{t-6} \\ & + \varepsilon_t + \phi_1 \varepsilon_{t-1} + (\phi_1^2 + \phi_2)\varepsilon_{t-2} + (\phi_1^3 + 2\phi_1\phi_2)\varepsilon_{t-3} + (\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2)\varepsilon_{t-4} \end{aligned}$$

where

$$\begin{aligned} A_5 &= \phi_1 A_4 + B_4 = \phi_1(\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2) + (\phi_1^3\phi_2 + 2\phi_1\phi_2^2) \\ &= \phi_1^5 + 4\phi_1^3\phi_2 + 3\phi_1\phi_2^2 \\ B_5 &= \phi_2 A_4 = \phi_2(\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2) = \phi_1^4\phi_2 + 3\phi_1^2\phi_2^2 + \phi_2^3 \\ C_5 &= C_4 + A_4 = 1 + \phi_1 + \phi_1^2 + \phi_2 + \phi_1^3 + 2\phi_1\phi_2 + \phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2 \end{aligned}$$

Step 5 $\to$ Step 6: y_t in terms of y_{t-6} and y_{t-7}

Substitute $y_{t-5} = \mu + \phi_1 y_{t-6} + \phi_2 y_{t-7} + \varepsilon_{t-5}$:

$$y_t = \mu\, C_6 + A_6\, y_{t-6} + B_6\, y_{t-7} + \sum_{j=0}^{5} A_j\, \varepsilon_{t-j}$$

where

$$\begin{aligned} A_6 &= \phi_1 A_5 + B_5 = \phi_1(\phi_1^5 + 4\phi_1^3\phi_2 + 3\phi_1\phi_2^2) + (\phi_1^4\phi_2 + 3\phi_1^2\phi_2^2 + \phi_2^3) \\ &= \phi_1^6 + 5\phi_1^4\phi_2 + 6\phi_1^2\phi_2^2 + \phi_2^3 \\ B_6 &= \phi_2 A_5 = \phi_2(\phi_1^5 + 4\phi_1^3\phi_2 + 3\phi_1\phi_2^2) = \phi_1^5\phi_2 + 4\phi_1^3\phi_2^2 + 3\phi_1\phi_2^3 \\ C_6 &= C_5 + A_5 \end{aligned}$$

The ε coefficients are $(A_0, A_1, \ldots, A_5) = (1,\ \phi_1,\ \phi_1^2 + \phi_2,\ \phi_1^3 + 2\phi_1\phi_2,\ \phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2,\ A_5)$.

Summary: General Pattern of A_k Coefficients

The A_k coefficients follow the combinatorial pattern (binomial in ϕ_1^2 and ϕ_2):

k	A_k
1	ϕ_1
2	$\phi_1^2 + \phi_2$
3	$\phi_1^3 + 2\phi_1\phi_2$
4	$\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2$
5	$\phi_1^5 + 4\phi_1^3\phi_2 + 3\phi_1\phi_2^2$
6	$\phi_1^6 + 5\phi_1^4\phi_2 + 6\phi_1^2\phi_2^2 + \phi_2^3$

Observation

The binomial coefficients $\binom{k-j-1}{j}$ appear in the expansion $A_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k-j-1}{j-1} \phi_1^{k-2j}\phi_2^j$, connecting the AR(2) dynamics to Pascal's triangle.

Check of Handwritten Derivations

All steps in the handwritten notes are correct. The key identities verified:

Step	A_k (written)	A_k (computed)	Status
$k=2$	$\phi_1^2+\phi_2$	$\phi_1^2+\phi_2$	✓
$k=3$	$\phi_1^3+2\phi_1\phi_2$	$\phi_1^3+2\phi_1\phi_2$	✓
$k=4$	$\phi_1^4+3\phi_1^2\phi_2+\phi_2^2$	$\phi_1^4+3\phi_1^2\phi_2+\phi_2^2$	✓

Step	B_k (written)	B_k (computed)	Status
$k=2$	$\phi_1\phi_2$	$\phi_1\phi_2$	✓
$k=3$	$\phi_1^2\phi_2+\phi_2^2$	$\phi_1^2\phi_2+\phi_2^2$	✓
$k=4$	$\phi_1^3\phi_2+2\phi_1\phi_2^2$	$\phi_1^3\phi_2+2\phi_1\phi_2^2$	✓

The μ accumulation and ε coefficients also match throughout. ✓

Conditional Mean and Variance at Each Step

Given the expansion $y_t = \mu\, C_k + A_k\, y_{t-k} + B_k\, y_{t-k-1} + \sum_{j=0}^{k-1} A_j\, \varepsilon_{t-j}$:

$$E(y_t \mid y_{t-k}, y_{t-k-1}) = \mu\, C_k + A_k\, y_{t-k} + B_k\, y_{t-k-1}$$

$$V(y_t \mid y_{t-k}, y_{t-k-1}) = \sigma^2 \sum_{j=0}^{k-1} A_j^2$$

Explicit variance at each step

$$\begin{aligned}
k = 1{:} &\quad \sigma^2 \\
k = 2{:} &\quad \sigma^2(1 + \phi_1^2) \\
k = 3{:} &\quad \sigma^2\big(1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2\big) \\
k = 4{:} &\quad \sigma^2\big(1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2 + (\phi_1^3 + 2\phi_1\phi_2)^2\big)
\end{aligned}$$

Summary

- The AR(2) model $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$ can be recursively expanded.
- After k substitutions: $y_t = \mu C_k + A_k y_{t-k} + B_k y_{t-k-1} + \sum_{j=0}^{k-1} A_j \varepsilon_{t-j}$.
- The recursion $A_{k+1} = \phi_1 A_k + B_k$, $B_{k+1} = \phi_2 A_k$ generates all coefficients.
- Binomial-type coefficients appear in A_k, connecting to Pascal's triangle.
- All handwritten derivations through $k = 4$ are verified correct.
- Extensions to $k = 5$ (y_{t-5}, y_{t-6}) and $k = 6$ (y_{t-6}, y_{t-7}) are derived here.
- The conditional variance grows with k: forecasting further back carries more uncertainty.