petr beckmann

{a history of}

π

PI

3.14159265358979323846
62643383279502884197169
39937510582097494459230781
16406286208998628034825342117
0679821480865132823066470938446
0955058223172535940812848111745028410
2701938521105559644622948954930381964428810
75665933446128475648233786783165271201909145648566
9234603486104543266482133936072602491412737245870066063155881748

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The Monte Carlo Method

Tremblez, ennemis de la France,
Rois ivres de sang et d'orgeuil!
Le peuple souverain s'avance,
Tyrans, descendez au cerceuil! [77]
French revolutionary song
of Lazare Carnot's time.

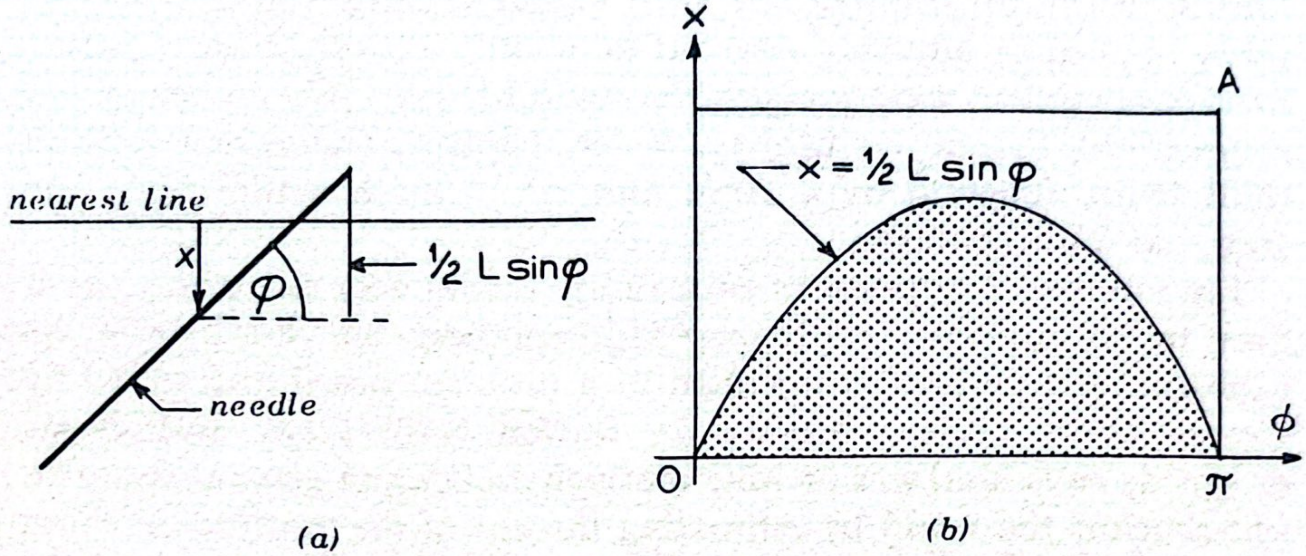
PROBABILITY theory is the mathematics of the 20th century. Its history goes back to the 16th century, but not until the present century did physicists and engineers fully realize that nature and the real world can be described exhaustively only by the laws governing their randomness. What physicists had considered exact until relatively recently, turned out to be merely the mean value of a much more impressive structure; and mean values can be very misleading. ("Put one foot in an ice bucket, and the other in boiling water; then on the average you will be comfortable.") Strange to relate, even as brilliant and recent a physicist as Albert Einstein regarded the probabilistic laws of quantum mechanics as testimony to our ignorance rather than as a valid description of the laws of nature.

The beginnings of probability theory go back to the *Liber de ludo aleae* (The book of games of chance), written about 1526 by Gerolamo Cardano (1501-1576), though not published until 1663. Cardano, of cubic equation fame (p. 90), was not only a mathematician, engineer and physician, but also a passionate gambler. Until the advent of the kinetic theory of gases in the 19th century, probability theory was rarely applied to anything else but gambling. The main contributors to its development were Jacques Bernoulli I (1654-1705, author of *Ars conjectandi*), Pascal, De Moivre, Euler, Laplace, Gauss and Poisson (1781-1840), followed by a large number of mathematicians in the 19th and 20th centuries.

The number π appears in probability theory very frequently, as it does in all branches of higher mathematics; but nowhere is its appearance more fascinating than in a problem posed and solved by George Louis Leclerc, Comte de Buffon (1707-1788). Buffon (as everybody calls him) was an able mathematician and general scientist, who shocked the world by estimating the age of the earth to be about 75,000 years, although every educated person in the 18th century knew that it was no older than about 6,000 years. Among his exploits is a test of one of Archimedes' supposed engines of war used in the defense of Syracuse. As told by Plutarch, the story includes a plausible description of the action of Archimedes' cranes and missile throwers, but by the Middle Ages, it had grown into a much exaggerated legend, and the *Book of Histories* by the Byzantine author John Tzetzes (ca. 1120-1183) repeats the story with many embellishments, such as the statement that Archimedes had burned the Roman ships to ashes at a distance of a bow shot by focusing the sun's beams onto the Roman fleet. The story (which is not contained in Plutarch's description) has persisted in many books down to our own days. Buffon, a man of considerable means and spare time, decided to test the feasibility of such a machine. Using 168 flat mirrors six by eight inches in an adjustable framework, he was able to ignite wooden planks at a distance of 150 feet, and he satisfied himself that Archimedes' alleged exploit was feasible. He did not, however, satisfy posterity, since the Syracusans would hardly have had the same leisure to focus 168 beams, nor would the Roman ships floating on the sea have held as still as Buffon's beams on the ground.

But back to Buffon's problem involving π. The problem which he posed (and solved) in 1777 was the following: Let a needle of length L be thrown at random onto a horizontal plane ruled with parallel straight lines spaced by a distance d (greater than L) from each other. What is the probability that the needle will intersect one of these lines?

We assume that "at random" means that any position (of the center) and any orientation of the needle are equally probable and that these two random variables are independent. Let the distance of



Buffon's problem.

the center of the needle from the nearest line be x, and let its orientation be given by ϕ (figure *a*). Since x is measured from the *nearest* line, we need only consider a single line, because the others involve only repetition of the same solution.

It is obvious from the figure that the needle will intersect a line if and only if

$$x < \tfrac{1}{2} L \sin \phi . \tag{1}$$

The problem is therefore equivalent to finding the probability

$$P(x < \tfrac{1}{2} L \sin \phi).$$

To find this probability, use the plane of rectangular coordinates x, ϕ, and consider the interior of the rectangle *OA* (figure *b*) whose points satisfy the inequalities

$$\begin{aligned} 0 < x < d/2 \\ 0 < \phi < \pi \end{aligned} \tag{3}$$

These are the intervals of possible values of x and ϕ, and therefore any point inside the rectangle *OA* corresponds to one and only one possible combination of position (x) and orientation (ϕ) of the needle. Since all such combinations are equiprobable, and the area of the rectangle represents the sum total of all possibilities that can arise (because, not quite beyond reproach, we regard this area as made up of all points inside it). However, not all of these possibilities will result in an intersection of the needle with a line; such an intersection, as we have found, will take place only under the condition (1), that is, for positions and orientations corresponding to points lying below the curve $x = ½L \sin \phi$ in figure *b*, so that the sum total of possibilities resulting in the intersection by the needle is given by the area under this curve. If, then, probability is the ratio of the number of favorable, to the number of possible, events under given conditions, the probability of intersection is given by the ratio of the shaded part to the entire rectangle *OA* in figure *b*, that is, the required probability (2) is

$$P = ½L \int_0^\pi \sin\phi \, d\phi \; : \; \pi d/2 = 2L/\pi d \qquad (4)$$

This is the result Buffon derived. He also attempted an experimental verification of his result by throwing a needle many times onto ruled paper and observing the fraction of intersections out of all throws. Whether he modified his result for an evaluation of π I do not know, but the problem and its solution were largely forgotten for the next 35 years, until one of the great mathematicians with whom France has been blessed, called attention to it and gave it a new twist.

Pierre Simon Laplace (1749-1827) was one of the "three great L's" among French mathematicians of the time. The other two, Joseph Louis Lagrange (1736-1813) and Adrien Marie Legendre (1752-1833), were his contemporaries, and all three survived the French Revolution as members of the Committee of Weights and Measures, which discarded the cubits, feet, pounds and miles of the old regime and worked out the metric system as we use it today. It was, incidentally, another mathematician, Lazare Carnot (1753-1823), who saved the young French republic in its hour of greatest need. Scared out of their wits by the cry for liberty, equality and fraternity, Europe's kings, princes, princelings, counts and whatnots turned on the Revolution. Threatened by internal confusion and the invading armies deep inside France, the Revolution seemed about to be crushed; but Carnot, member of the Committee for Public Safety in charge of military affairs, took command and sent the invaders packing on all fronts, becoming *organisateur de la victoire*, the hero of the French Revolution. But like so many other sincere revo-

JOSEPH LOUIS LAGRANGE
(1736-1813)

PIERRE SIMON LAPLACE
(1749-1827)

lutionaries after him, Carnot soon observed that a revolution only replaces one tyranny by another, and refusing to go along with its excesses, he was driven into exile as a "royalist." Significantly, his chair of geometry at the *Institut National* was unanimously voted to a general; a general by the name of Napoleon Bonaparte, another one in a long line of power-hungry careerists who was to preach liberty and practice oppression.

PIERRE Simon Laplace is known, above all, for authoring two masterpieces, *Mecanique céleste* (5 vols., 1799-1825) and *Théorie analytique des probabilites* (1812). The former was the greatest work on celestial mechanics since Newton's *Principia,* including many new mathematical techniques, such as the theory of potential. Asked by Napoleon why in the entire work on celestial mechanics he had not once mentioned God, Laplace replied, *Sire, je n'avais pas besoin de cette hypothèse* — Sire, I had no need of that hypothesis. Napoleon, incidentally, appointed Laplace Minister of Interior, but after six weeks dismissed him again, commenting that he "carried the spirit of the infinitely small into the management of affairs." The *Théorie analytique* is the foundation of modern probability theory. Among many new mathematical techniques it contains the integral transform that is today the daily bread of every systems engineer and analyst of electrical circuits.

It also contains a discussion of Buffon's problem, which Laplace saw in a new light. From the first and last expressions in (4) we have

$$\pi = 2L/dP \tag{5}$$

and this is an entirely new method of evaluating π: The length of the needle L and the spacing between the lines d are known (usually one makes $L = d$), and the probability of intersection P can be measured by throwing a needle onto ruled paper a very large number of times, recording the fraction of throws resulting in an intersection of the needle with a line.

This method, which Laplace generalized for paper with two sets of mutually perpendicular lines, has been used by several people as a playful diversion to calculate the first decimal places of π by thousands of throws. One of them was a certain Captain Fox, who indulged in this sport while recovering from wounds incurred in the American Civil War.[78]

It is not difficult to calculate the probability of obtaining π correct to k decimal places in N throws.[79] The results of such a calculation show that this method is very inefficient as far as the numerical computation of π is concerned; for example, the probability of obtaining π correct to 5 decimal places in 3,400 throws of the needle is less than 1.5%, which is very poor.

Nevertheless, Laplace had discovered a powerful method of computation that did not come into its own until the advent of the electronic computer. The method that Laplace proposed consists in finding a numerical value by realizing a random event many times and observing its outcome experimentally. This is today known as a Monte Carlo method (Monte Carlo is the European Las Vegas), and it is used in a wide field of applications ranging from economics to nuclear physics.

Let us first take the example of calculating π by this method. A computer can easily throw a needle 500 times a second, or 1.8 million times per hour. Not literally, of course, but it can be programmed to select a random number (x) for the position of the needle and another (ϕ) for its orientation, which is just as good; it *simulates* the throwing of a needle. It can also be programmed to observe whether the needle has intersected or not, that is, whether the inequality (1) is satisfied or not. Finally, it is programmed to record the number of intersections in the total number of throws, and after computing the resulting value of π by formula (5), to print the value it has found.

A program of this type is shown on the next page. It is written in BASIC, a simple, but powerful computer language.

```
10 LET N = 0
20 PRINT 'NO. OF THROWS', 'PI'
30 FOR J = 1 TO 24
40 FOR K = 1 TO 500
50 LET X = RND(X)
60 LET U = SIN(3.1415927*RND(F))
70 IF X GT U THEN 90
80 LET N = N + 1
90 NEXT K
100 LET T = 500*J
110 LET P = 2*T/N
120 PRINT T, P
130 NEXT J
140 END
```

The program was actually run[80] and resulted in the following values of π in the first 12,000 throws:

NO. OF THROWS	PI
500	3.2154341
1000	3.2414911
1500	3.1645569
2000	3.1620553
2500	3.1407035
3000	3.1430068
3500	3.1460674
4000	3.1421838
4500	3.1435557
5000	3.1446541
5500	3.1401656
6000	3.1217482
6500	3.1175060
7000	3.1354983
7500	3.1453135
8000	3.1452722
8500	3.1493145
9000	3.1441048
9500	3.1384209
10000	3.1392246
10500	3.1493701
11000	3.1527658
11500	3.1485284
12000	3.1417725

We cannot in this way, of course, obtain a better value of π than the one we inserted in line 60 of the program on the opposite page (it occurs there to make the orientation of the needle uniformly distributed between 0° and 180°), and the same line results in an error owing to a series of successive roundings off. However, even if these technicalities were corrected, the result would still be poor, as predicted by the calculation of the probability of obtaining k correct decimal places in a series of n throws. In the same processing time (53 seconds) we could have obtained a much better value, for example, by Euler's method.

But if the method is not very efficient for calculating π, it is very powerful in other applications. Suppose, for example, that we wish to calculate the mean value of a complicated function of a random variable. This is found by an integration involving the probability density function of the random variable. But sometimes the resulting integral is so complicated that it takes a long time to write the program and that it involves a costly amount of processing time. In that case we do not program the computer for the complicated evaluation of the integral, but we make it simulate the random variable and its function and we make it compute the arithmetic mean of, say, one hundred thousand trials. The result is the required mean value.

Or suppose we wish to find a complicated multiple integral. A Monte Carlo method of finding it (instead of writing a cumbersome program) is to let the computer "shoot" n-tuplets of random numbers. These represent a coordinate in (n-dimensional) space and the coordinate either lies in the volume determined by the integral ("hit") or it does not ("miss"). Then we let the computer shoot at the target, say, half a million times. The number of hits is then proportional to the n-tuple integral.

The man who taught us to program electronic computers in this way was Pierre Simon Laplace. His computer was neither electronic nor digital. It was an analog computer consisting of one needle and one piece of ruled paper.

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