

# MCMC and particle filtering for dynamic INAR processes

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In this article, we present a dynamic version of the integer autoregressive (INAR) processes for count data. The proposed Bayesian model provides a unification of the previously considered models to describe temporal correlations in univariate time series of counts. We develop Bayesian inference for the proposed class of models via MCMC and introduce a particle filtering (PF) algorithm for sequential inference. The new class of models are compared with their static counter parts using actual count series and additional insights provided by the new models are discussed.

## KEYWORDS

Count data, MCMC, sequential Monte Carlo, Bayesian filtering and prediction

## 1 | INTRODUCTION AND OVERVIEW

Time series of counts arise in many areas such as business, economics, engineering, and medicine. Applications include, among others, the modeling of the number of deaths from a specific disease in a given month [see Schmidt and Pereira (2011)], the number of arrivals to a call center [see Aktekin and Soyer (2011)], the number of monthly mortgage defaults [see Aktekin et al. (2013)], time series data on crash counts in different regions [Hu et al. (2013)], the number of accidents in a given time interval [Serhiyenko et al. (2014)], the number of weekly shopping trips of households [see Aktekin et al.(2018)], and network flows [see Chen et al. (2019)]. Other applications of discrete-valued time series can be found in the volume by Davis et al. (2016). The analysis of discrete-valued time series poses both methodological and computational challenges. Davis et al. (2021) provide an overview of models for time series count data and discuss methodological issues. Recent advances in Bayesian modeling of count time series are presented in Soyer and Zhang (2021), and computational issues are discussed. Silva et al. (2023) developed a Bayesian framework for count time series under censoring, which Silva et al. (2025) recently extended to series with missing observations. In

a related vein, Silva Pereira and McCabe (2019) proposed Bayesian outlier detection methods for non-Gaussian time series, including counts.

### **INAR processes.**

An important class of models for time series count data is integer autoregressive (INAR) processes. They belong to the class of *observation-driven* time series models in the general framework of Cox (1981). The first order Poisson INAR process was originally introduced by McKenzie (1985) and its properties and extensions were discussed in later articles by Al-Osh and Alzaid (1987) and by McKenzie (1988). The recent literature on INAR and related processes is vast. For instance, Weiß (2015) consider INAR(1) processes with serially dependent innovations, while Lopez *et al.* (2025) extend the INAR processes to allow generalized Katz innovations. Monteiro *et al.* (2012) extends INAR processes to self-exciting threshold autoregressive processes, with a Bayesian approach via Markov chain Monte Carlo (MCMC) sampling provided by Yang *et al.* (2022).

Bayesian inference for the INAR(1) model with Poisson errors was considered by Silva *et al.* (2005) using conjugate priors and a Gibbs sampler with a Metropolis step. Neal and Subba Rao (2007) discussed Bayesian inference for integer ARMA processes with Poisson errors via MCMC. In a recent paper, Marques *et al.* (2022) proposed a data augmentation step that allowed the implementation of a Gibbs sampler with conjugate full posterior conditionals. The authors also considered generalizations of INAR(1) by using a finite mixture model for the error process to account for potential overdispersion in time series, as well as a hierarchical extension of the Poisson error INAR(1) model for time-varying arrival rates. The latter is based on a Dirichlet process (DP) prior to the unknown distribution of dynamic arrival rates. The use of the DP prior implies exchangeability of the time-varying arrival rates and allows for clustering of homogeneous arrival rates.

### **Contributions.**

In this paper, we revisit the work of Marques *et al.* (2022) and propose a Markovian evolution for the time-varying arrival rates as an alternative to the exchangeability implied by their Dirichlet process prior. The proposed structure yields a new class of INAR processes and, by adopting a Bayesian perspective, unifies the observation-driven INAR processes with the parameter-driven state-space models of Aktekin *et al.* (2013) for count time series; it is, to the best of our knowledge, the first state-space version of INAR processes in the sense of West and Harrison (1997). Second, the conditional conjugacy implied by the model enables full Bayesian analysis via Gibbs sampling. Third, we develop a particle filter for sequential Bayesian inference and prediction. To the best of our knowledge, this is a tool that has not been previously used for INAR processes. Finally, we extend the model and methodology to INAR( $p$ ) processes. Beyond its methodological contributions, the paper also serves as an illustration of applied Bayesian time-series analysis, with particular emphasis on probabilistic inference, sequential learning and forecasting, and dynamic model comparison.

A synopsis of our paper is as follows. In Section 2, we start with a brief introduction to the INAR(1) process with Poisson errors and present our dynamic model. In doing so, we discuss the Markov process for the time-varying arrival rates and discuss its implications on the INAR(1) model. We illustrate how the model provides a unification of the observation- and parameter-driven processes considered for time series of counts, as well as its ability to deal with overdispersed data. Section 3 develops a Bayesian analysis of the model. A Gibbs sampler using data augmentation and a particle filter for sequential analysis are presented. An extension to the  $p$ -th order dynamic INAR process is discussed in Section 4 and Bayesian inference using Gibbs sampler and particle filtering is developed. We illustrate an implementation of the proposed model using actual data in Section 5. We discuss what types of additional insight can be obtained from the model and the Bayesian analysis. We conclude with some final remarks in Section 6.

## 2 | THE DYNAMIC INAR(1) MODEL

### 2.1 | A brief review of the INAR(1) model

For a stationary time-series of counts  $Y_t$ , the INAR(1) model is defined as

$$Y_t = \alpha \circ Y_{t-1} + \varepsilon_t \quad (1)$$

where “ $\circ$ ” is the binomial thinning operation represented by

$$\alpha \circ Y_{t-1} = \sum_{j=1}^{Y_{t-1}} B_{jt} \quad (2)$$

where  $B_{jt}$  are Bernoulli random variables with probability  $\alpha$ . If we define  $M_t = \alpha \circ Y_{t-1}$  in (2.2), then conditionally on  $Y_{t-1}$  and  $\alpha$ ,  $M_t$  has a binomial distribution with parameters  $\alpha$  and  $Y_{t-1}$ . In the above setup, it is important to note that the  $B_{jt}$ 's are independent of the  $\varepsilon_t$ 's and that the thinning is performed at each time point  $t$  independently of the thinning at other periods. As pointed out by Kedem and Fokianos (2002, Chapter 5), the INAR(1) process can be interpreted as a special case of branching processes with immigration where  $Y_t$  is the population at time  $t$  which consists of two components:  $M_t$ , those who survive from time  $(t-1)$  with probability  $\alpha$  and  $\varepsilon_t$ , those who arrive at the beginning of time  $t$ .

If  $\{\varepsilon_t\}$  is a sequence of i.i.d. Poisson random variables with rate  $\theta$  in (1), then the model is referred to as an INAR(1) process with Poisson errors. Alternatively, the error distribution can be assumed as geometric or negative binomial as discussed in Weiß (2008) or one can consider a mixture of Poisson and geometric as in Marques *et al.* (2022). In our development, we consider the INAR processes with Poisson errors.

If  $Y_0$  is assumed to be Poisson with rate  $\theta/(1-\alpha)$ , then it can be shown that  $Y_t$  is a stationary Poisson series with parameter  $\theta/(1-\alpha)$ . In addition, the autocorrelation function of INAR(1) processes is given by  $\rho_Y(k) = \alpha^k$  for  $k > 0$ ; see McKenzie (1988). In fact, the behavior of the autocorrelation function is not limited to the Poisson errors, but it holds for all INAR(1) models.

Given  $Y_{t-1}$ , the conditional distribution of  $Y_t$  is obtained as a convolution of a binomial and a Poisson given by

$$p(Y_t | Y_{t-1}, \theta, \alpha) = \sum_{j=0}^{\min(Y_{t-1}, Y_t)} \frac{e^{-\theta} \theta^{Y_t-j}}{(Y_t-j)!} \binom{Y_{t-1}}{j} \alpha^j (1-\alpha)^{Y_{t-1}-j}, \quad (3)$$

with  $E[Y_t | Y_{t-1}, \theta, \alpha] = \alpha Y_{t-1} + \theta$ . The forecast distribution for  $h$ -steps ahead,  $p(Y_{t+h} | Y_t, \theta, \alpha)$ , can be obtained as well as the conditional mean

$$E[Y_{t+h} | Y_t, \theta, \alpha] = \alpha^h \left( Y_t - \frac{\theta}{(1-\alpha)} \right) + \frac{\theta}{(1-\alpha)}. \quad (4)$$

It is important to note that the forecast distributions  $h$ -steps ahead are not available in closed form with more general error distributions, in which case numerical computations are necessary (see Weiß, 2018, Section 2.6).

## 2.2 | INAR(1) model with dynamic arrival rates

An earlier state-dependent parameter generalization of INAR models was considered in Thyregod et al. (1999) who proposed a self-exciting threshold-based (SET) INAR process for analysis of tipping bucket rainfall measurements. Monteiro et al. (2012) showed the stationarity of the process and investigated asymptotic properties of the maximum likelihood estimators. As previously mentioned Yang et al. (2022) developed Bayesian analysis for the for the SET-INAR processes using MCMC methods and Lopez et al. (2025) considered an extension of SET-INAR processes using generalized Katz innovations. For modeling state-dependent parameters, the SET-INAR approach is considered by many as the standard in count time-series community.

As an alternative to the existing approaches in the literature, we follow the standard approach in state space modeling for count time-series (West and Harrison, 1997) and consider a generalization of the INAR(1) process with Poisson errors in (1) by allowing the Poisson rates of  $\epsilon_t$ 's to be time-varying, i.e.,  $\theta_t$ . In this case, the observation model can be written as

$$Y_t - M_t | M_t, \theta_t \sim \text{Poisson}(\theta_t) I(Y_t \geq M_t), \quad (5)$$

which is a shifted Poisson for  $Y_t$  with rate  $\theta_t$ . We also have

$$M_t | Y_{t-1}, \alpha \sim \text{Bin}(Y_{t-1}, \alpha).$$

We assume that  $\theta_t$  has a Markov evolution characterized by

$$\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)} \sim \text{Beta}(\gamma a_{t-1}, (1 - \gamma) a_{t-1}, \theta_{t-1} / \gamma), \quad (6)$$

which is a scaled beta distribution on  $(0, \theta_{t-1} / \gamma)$  and can be rewritten as a system equation

$$\theta_t = \frac{\theta_{t-1}}{\gamma} w_t \quad (7)$$

where  $w_t | Y^{(t-1)}, M^{(t-1)} \sim \text{Beta}[\gamma a_{t-1}, (1 - \gamma) a_{t-1}]$ .

The quantity  $0 < \gamma < 1$  is a discount factor in the sense of West and Harrison (1997),  $Y^{(t)} = (Y_t, Y^{(t-1)})$  and  $M^{(t)} = (M_t, M^{(t-1)})$ . In the system equation (7),  $w_t$  can be considered as a system error term with mean  $\gamma$  and (7) provides a random walk-type evolution for  $\theta_t$ . The Markov evolution (7) was originally introduced by Smith and Miller (1986) who analyzed exponential observation models; although their approach was not fully Bayesian, Harvey and Fernandez (1989) showed that the same structure can be used for Poisson arrival rates; More recently, (7) was used in Gamerman et al. (2013) for parameter driven non-Gaussian time series.

In our proposed dynamic INAR(1) process, Equations (5) and (7) will play the roles of the observation equation and the system equation, respectively. We complete the definition of our model by assuming that, at time  $t - 1$ , the rate  $\theta_{t-1}$  follows a gamma distribution

$$\theta_{t-1} | M^{(t-1)}, Y^{(t-1)} \sim \text{Gamma}(a_{t-1}, b_{t-1}). \quad (8)$$

It follows from (6) and (8) that the forecast distribution of  $\theta_t$  at  $t - 1$  is given by

$$\theta_t | M^{(t-1)}, Y^{(t-1)} \sim \text{Gamma}(\gamma a_{t-1}, \gamma b_{t-1}). \quad (9)$$

Using the Bayes' rule

$$p(\theta_t | M^{(t)}, Y^{(t)}) \propto p(\theta_t | M^{(t-1)}, Y^{(t-1)}) p(Y_t - M_t | M_t, \theta_t),$$

it can be shown that the posterior (or filtering) distribution of  $\theta_t$  can be obtained as

$$\theta_t | M^{(t)}, Y^{(t)} \sim \text{Gamma}(a_t, b_t), \quad (10)$$

for  $a_t = \gamma a_{t-1} + (Y_t - M_t)$  and  $b_t = \gamma b_{t-1} + 1$ .

We note that starting at time 0 with prior  $\theta_0 \sim \text{Gamma}(a_0, b_0)$ , the proposed model provides us with a conjugate update of the dynamic rates which is attractive in developing posterior and predictive Bayesian inferences, as will be discussed in Section 3. Other attractive features of the model are its ability to deal with overdispersed time series of counts and the inclusion of some of the previously considered models as special cases. We will elaborate on this in Section 2.3.

### INAR(1) with dynamic thinning and covariates.

An alternative dynamic INAR(1) model can be considered by assuming a dynamic structure for the thinning probabilities in (1). More specifically, we can write a dynamic version of (1) as

$$Y_t = \alpha_t \circ Y_{t-1} + \epsilon_t$$

where  $\epsilon_t$ 's are Poisson error terms with static rate  $\theta$  and the binomial thinning is now given by

$$M_t \equiv \alpha_t \circ Y_{t-1} = \sum_{j=1}^{Y_{t-1}} B_{jt}$$

where  $B_{jt}$ 's are conditionally independent Bernoulli random variables with probability  $\alpha_t$ . Time evolution of  $\alpha_t$ ' can be described by using a logit type transformation and a Markovian structure such as

$$\phi_t = \log\left(\frac{\alpha_t}{1 - \alpha_t}\right) = \phi_{t-1} + w_t, \quad (11)$$

where  $w_t$ 's are independent zero mean Gaussian innovation terms with variance  $\sigma_w^2$ . Bayesian analysis of the model using a Gibbs sampler will require use of some data augmentation type approach within the Gibbs as in Polson et al. (2013). This is currently under investigation.

It is also possible to incorporate covariates into the dynamic INAR(1) models. For example, we can define  $\lambda_t$  as the dynamic arrival rate of the error term  $\epsilon_t$  in (5) via

$$\lambda_t = \theta_t \exp\{\beta' z_t\},$$

where  $z_t$  is a vector of covariates with unknown parameters  $\beta$  and  $\theta_t$  is the baseline arrival rate following a Markov evolution as in (6). As noted by Aktekin et al. (2013), Bayesian analysis of the model will require use of a Metropolis step within the Gibbs sampler discussed in Section 3.1.

## 2.3 | Other properties of the dynamic INAR(1) model

The Markov evolution given by (6) enables us to deal with overdispersed data. Using the forecast (or prior) distribution of  $\theta_t$  given by (9), we can obtain the predictive distribution of  $\epsilon_t = Y_t - M_t$  via

$$p(\epsilon_t | \gamma, M^{(t)}, Y^{(t-1)}) = \int_0^\infty p(\epsilon_t | \theta_t, M_t) p(\theta_t | \gamma, M^{(t)}, Y^{(t-1)}) d\theta_t,$$

which can be reduced to

$$p(\epsilon_t | \gamma, M^{(t)}, Y^{(t-1)}) = \frac{\Gamma(\gamma a_{t-1} + \epsilon_t)}{\Gamma(\epsilon_t + 1) \Gamma(\gamma a_{t-1})} \left(1 - \frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{\epsilon_t} \left(\frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{\gamma a_{t-1}}, \quad (12)$$

which is a negative binomial for integer values of  $\gamma a_{t-1}$ . As a result, the Markov structure provides with a negative binomial type dynamic error distribution which can handle over-dispersed counts. As will be discussed in Section 3, a Bayesian analysis using this model can also be developed.

### Unifying the count time series models.

The dynamic INAR(1) model has two static parameters: the thinning probability  $\alpha$  and the discount factor  $\gamma$ . These two parameters play an important role in assessing the suitability of different classes of models for analyzing time series count data. We note that as  $\alpha \rightarrow 0$  in the dynamic INAR(1) model, the autocorrelation of  $Y_t$ 's disappears and in the limit  $\alpha = 0$  the model reduces to a parameter-driven model as in Harvey and Fernandes (1989) and Aktekin and Soyer (2011) where, given  $\theta_t$ 's,  $Y_t$ 's are conditionally independent Poisson counts. On the other hand, as we can see from the Markov model for  $\theta_t$  and the system equation (7), the arrival rates become smoother over time as  $\gamma \rightarrow 1$ . In the limit  $\gamma = 1$ ,  $\theta_t = \theta$  for all  $t$  and our model reduces to the static INAR(1) process of equation (1). In other words, these two cases can be obtained in our dynamic INAR(1) by putting a degenerate prior for  $\alpha$  in 0 or for  $\gamma$  in 1. The trivial case of a Poisson time series of white noise can be obtained by placing a joint degenerate prior for  $(\alpha, \gamma)$  in  $(0, 1)$ . Therefore, the proposed dynamic INAR(1) process provides a unification of these models.

## 3 | MONTE CARLO-BASED POSTERIOR INFERENCE

In this section, we propose an MCMC scheme for joint posterior inference of all unknowns of the model. We also derive a particle filter for online inference. In our development, we assume independent priors for model parameters  $\alpha$  and  $\gamma$ . Specifically, we use a beta prior for  $\alpha$  as  $\alpha \sim \text{Beta}(a_\alpha, b_\alpha)$  and denote the prior for the discount parameter by  $p(\gamma)$ . Furthermore,  $\alpha$  and  $\gamma$  are a priori assumed to be independent of the initial arrival rate  $\theta_0$ .

### 3.1 | Gibbs sampler

We can develop Bayesian inference using a Gibbs sampler given counts  $Y^{(t)}$  from  $T$  time periods. In doing so, we need the posterior full conditionals of  $\alpha, \gamma, \theta^{(T)} = (\theta_1, \dots, \theta_T)$ , and  $M^{(T)} = (M_1, \dots, M_T)$ . Among these, the trickiest

is the full conditional distribution of  $M^{(T)}, p(M^{(T)}|\alpha, \gamma, \theta^{(T)}, Y^{(T)})$ .

### Learning $M^{(T)}$ .

Given  $\alpha$  and  $\gamma$ , suppressed below, assuming  $M_1 = 0$ , we can write the joint distribution  $p(\theta_1, Y_1, \theta_2, Y_2, M_2, \dots, \theta_T, Y_T, M_T)$  as

$$p(\theta_1|D_0)p(Y_1|\theta_1)\prod_{t=2}^T p(\theta_t|\theta_{t-1}, M^{(t-1)}, Y^{(t-1)})p(M_t|Y_{t-1})p(Y_t|M_t, \theta_t), \quad (13)$$

so the full conditional for each  $M_t$  can be obtained as

$$p(M_t|M^{(-t)}, \theta^{(T)}, Y^{(T)}) \propto \left\{ \prod_{s=t+1}^T p(\theta_s|\theta_{s-1}, M^{(s-1)}, Y^{(s-1)}) \right\} \\ \times p(M_t|Y_{t-1})p(Y_t|M_t, \theta_t), \quad (14)$$

for  $t = 2, \dots, T-1$ , where  $M^{(-t)} = \{M_s; s \neq t\}$  and  $D_0$  denotes the prior information. It is important to note that in the above product

$$(\theta_s|\theta_{s-1}, M^{(s-1)}, Y^{(s-1)}) \sim \text{Beta}[\gamma a_{s-1}, (1-\gamma)a_{s-1}]I(\theta_s < \theta_{s-1}/\gamma). \quad (15)$$

This suggests that for  $s = (t+1), (t+2), \dots, T$ , the  $a_{s-1}$  values should be evaluated at the given value of  $M_t$ . For example, when  $s = (t+1)$ , we have  $a_t = \gamma a_{t-1} + (Y_t - M_t)$ . Therefore, for each value of  $M_t$ , we need to evaluate  $a_t, a_{t+1}, \dots, a_T$  and these values will be different than the values in updating of  $\theta_t$ 's. For  $t = T$ , we can write

$$p(M_T|M^{(-T)}, \theta^{(T)}, Y^{(T)}) \propto p(M_T|Y_{T-1})p(Y_T|M_T, \theta_T).$$

### Learning $\alpha$ .

The full conditional of  $\alpha$  can be obtained as

$$\alpha|M^{(T)}, Y^{(T)} \sim \text{Beta}\left(a_\alpha + \sum_{t=2}^T M_t, b_\alpha + \sum_{t=2}^T (Y_{t-1} - M_t)\right). \quad (16)$$

### Learning $\theta^{(T)}$ .

In order to implement a Gibbs sampler, the full conditional of the vector  $\theta^{(T)}, p(\theta_1, \dots, \theta_T|M^{(T)}, \gamma, Y^{(T)})$ , can be obtained by using the forward filtering backward sampling (FFBS) algorithm of Frühwirth-Schnatter (1994). We can write the joint full conditional as

$$p(\theta_T|M^{(T)}, Y^{(T)}, \gamma)p(\theta_{T-1}|\theta_T, M^{(T-1)}, Y^{(T-1)}, \gamma) \cdots p(\theta_1|\theta_2, Y_1, \gamma), \quad (17)$$

where  $p(\theta_{t-1}|\theta_t, M^{(t-1)}, Y^{(t-1)}, \gamma)$  is a shifted-gamma density over  $(\gamma\theta_t, \infty)$ , denoted as

$$\text{Gamma}((1-\gamma)a_{t-1}, b_{t-1})I(\theta_{t-1} > \gamma\theta_t).$$

### Learning $\gamma$ .

Note that to obtain the full conditional of  $\gamma$  we can take the joint distribution in (13) and note that  $p(\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)})$  is conditional on  $\gamma$ . Thus, the full conditional of  $\gamma$  can be obtained as

$$p(\gamma | \theta^{(T)}, M^{(T)}, Y^{(T)}) \propto \left\{ \prod_{t=2}^T p(\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)}, \gamma) \right\} p(\gamma), \quad (18)$$

where  $p(\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)}, \gamma)$  is the scaled beta distribution given by (6). We can use a Metropolis step or use a discrete prior to drawing samples from the full conditional of  $\gamma$ .

Alternatively, we can learn about  $\gamma$  outside the Gibbs sampler by obtaining its marginal likelihood. The marginal likelihood for the discount parameter  $\gamma$ , conditional on  $M^{(T)}$ , is

$$p(\epsilon_1, \dots, \epsilon_T | M^{(T)}, Y^{(T)}, \gamma) = \prod_{s=1}^T p(\epsilon_s | M^{(s)}, Y^{(s-1)}, \gamma), \quad (19)$$

where  $p(\epsilon_s | M^{(s)}, Y^{(s-1)}, \gamma)$  is given by the predictive distribution (12). This can also be used to obtain the posterior distribution of  $\gamma$ , that is,

$$p(\gamma | Y^{(T)}) \propto p(\epsilon_1, \dots, \epsilon_T | \gamma) p(\gamma),$$

where  $p(\epsilon_1, \dots, \epsilon_T | \gamma)$  can be obtained using samples generated from the distribution of  $p(M^{(T)} | \gamma, Y^{(T)})$  through the Gibbs sampler. One can also use a discrete prior in  $p(\gamma)$  to evaluate the posterior distribution of  $\gamma$ . In Algorithm 1, we show the schematic representation of the Gibbs sampler for our proposed dynamic INAR(1) model.

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#### Algorithm 1 Gibbs sampler algorithm

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- 1: Initial values  $(\theta^{(T)}, M^{(T)}, \alpha, \gamma)^{(0)}$  and repeat the following 4 steps iteratively.
- 2: **for**  $g = 1 \dots G$  **do**
- 3: Draw  $\alpha^g \sim \text{Beta}(a_\alpha + \sum_{t=2}^T M_t, b_\alpha + \sum_{t=2}^T (Y_{t-1} - M_t))$ .
- 4: Draw  $\theta_T^g \sim \text{Gamma}(a_T, b_T)$
- 5: **for**  $t = T - 1 \dots 1$  **do**

$$\text{Draw } \theta_t^g \sim \text{Gamma}((1 - \gamma)a_t, b_t) I(\theta_t > \gamma \theta_{t+1}^g)$$

- 6: **end for**
- 7: **for**  $t = 2 \dots T - 1$  **do**

Draw  $M_t^g \in \{0, 1, \dots, \min(Y_t, Y_{t-1})\}$  from the discrete distribution proportional to

$$\left\{ \prod_{s=t+1}^T p(\theta_s^g | \theta_{s-1}^g, M^{(s-1)}, Y^{(s-1)}) \right\} p(M_t | Y_{t-1}) p(Y_t | M_t, \theta_t^g).$$

- 8: **end for**
  - 9: **end for**
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### 3.2 | Sequential Monte Carlo and particle filtering

It is well known that MCMC methods are not computationally efficient for sequential Bayesian learning and forecasting, as they require rerunning of the Markov chains with each additional observation to obtain the posterior distributions of time-varying parameters, such as Poisson rates  $\theta_1, \dots, \theta_T$  in our case. Sequential Monte Carlo (SMC) methods have been proposed to alleviate such computational inefficiencies. An important class of SMC methods are particle filters (PF), that was originally proposed by Gordon et al. (1993); see Lopes and Tsay (2011) for a review of PF and Singpuwalla et al. (2018) for a historical perspective.

As pointed out by Carvalho et al. (2010), and extensively discussed by Lopes et al. (2011), learning about static parameters in PF is not trivial, but if the conditional posterior distributions of static parameters are available with known (conditional) sufficient statistics, one can develop an efficient recursive updating scheme which they referred to as *particle learning* (PL). We can develop a PL algorithm in the sense of Lopes et al. (2011) for the dynamic INAR(1) model.

Assume that in the time period  $t - 1$ , given the data  $Y^{(t-1)}$ , we have particles  $\{\theta_{t-1}^i, M_{t-1}^i, \alpha^i\}$ , for  $i = 1, \dots, N$ . Note that given this set of particles, we can obtain a sample from the prior  $p(M_t | Y^{(t-1)})$  by sampling  $M_t^i$  from the binomial distribution  $Bin(Y_{t-1}, \alpha^i)$ , for  $i = 1, \dots, N$ . Our objective is to resample  $\theta_{t-1}$  based on the observed value in time  $t$ ,  $Y_t$ . For this, we need resampling weights that are proportional to the predictive  $p(Y_t | M_t, \theta_{t-1}, M^{(t-1)}, Y^{(t-1)})$ . We can obtain the distribution as follows.

$$\int p(Y_t | M_t, \theta_{t-1}, \theta_t, M^{(t-1)}, Y^{(t-1)}) p(\theta_t | M_t, \theta_{t-1}, M^{(t-1)}, Y^{(t-1)}) d\theta_t,$$

which reduces to

$$p(Y_t | M_t, \theta_{t-1}, M^{(t-1)}, Y^{(t-1)}) = \int p(Y_t | M_t, \theta_t) p(\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)}) d\theta_t$$

where  $(Y_t | M_t, \theta_t) \sim Poi(\theta_t) I(Y_t \geq M_t)$  and  $(\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)})$  is a scaled beta distribution denoted as

$$(\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)}) \sim Beta(\gamma a_{t-1}, (1 - \gamma) a_{t-1}, \theta_{t-1} / \gamma).$$

It can be shown that  $p(Y_t | M_t, \theta_{t-1}, M^{(t-1)}, Y^{(t-1)})$  is a confluent hypergeometric negative binomial distribution that we can evaluate numerically. Specifically, we have  $p(Y_t | M_t, \theta_{t-1}, M^{(t-1)}, Y^{(t-1)})$  is given by

$$\int_0^{\theta_{t-1}/\gamma} \frac{\theta_t^{(Y_t - M_t)} e^{-\theta_t}}{(Y_t - M_t)!} \xi(\gamma, a_{t-1}) (\gamma / \theta_{t-1})^{a_{t-1} - 1} \theta_t^{\gamma a_{t-1} - 1} \left( \frac{\theta_{t-1}}{\gamma} - \theta_t \right)^{(1 - \gamma) a_{t-1} - 1} d\theta_t,$$

where  $\xi(\gamma, a_{t-1}) = Beta(\gamma a_{t-1}, (1 - \gamma) a_{t-1})$ . Recall that the Beta function is defined as  $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ .

Also, by using change of variable  $u_t = (\gamma / \theta_{t-1}) \theta_t$ , we can write the above integral as

$$\kappa(Y_t, M_t, \theta_{t-1}, \gamma, a_{t-1}) \int_0^1 e^{-\frac{\theta_{t-1}}{\gamma} u_t} u_t^{\gamma a_{t-1} + (Y_t - M_t) - 1} (1 - u_t)^{(1 - \gamma) a_{t-1} - 1} du_t,$$

where

$$\kappa(Y_t, M_t, \theta_{t-1}, \gamma, a_{t-1}) = \frac{(\theta_{t-1}/\gamma)^{(Y_t - M_t)}}{(Y_t - M_t)!} \xi(\gamma, a_{t-1}),$$

with the integral in the above expression represented as

$$Beta(Y_t - M_t + \gamma a_{t-1}, (1 - \gamma) a_{t-1}) CHF(a, a + b, -c),$$

where  $CHF(a, a + b, -c)$  is the confluent hyper-geometric function of Abramowitz and Stegun (1968) with  $a = (Y_t - M_t) + \gamma a_{t-1}$ ,  $b = (1 - \gamma) a_{t-1}$  and  $c = \theta_{t-1}/\gamma$ . To evaluate the CHF function, we can use the R package `gs1` of Hankin (2006). Thus, the weights  $p(Y_t | M_t^i, \theta_{t-1}^i, M^{(t-1)}, Y^{(t-1)})$  are proportional to

$$\begin{aligned} \omega^i &\propto \kappa(Y_t, M_t^i, \theta_{t-1}^i, \gamma, a_{t-1}) \\ &\times Beta(Y_t - M_t^i + \gamma a_{t-1}, (1 - \gamma) a_{t-1}) \\ &\times CHF((Y_t - M_t^i) + \gamma a_{t-1}, (Y_t - M_t^i) + a_{t-1}, -\theta_{t-1}^i/\gamma), \end{aligned} \quad (20)$$

which are used to obtain the resampled values of  $(M_t^{k(i)}, \theta_{t-1}^{k(i)})$ . Note that we have not resampled  $\alpha$  values, that is, their samples are based on the data from  $(t - 1)$ , since these are not needed to resample the values of  $(M_t^{k(i)}, \theta_{t-1}^{k(i)})$ . However, given  $M_t^{k(i)}$ , we can have resampling values of  $\alpha$  by updating enough statistics of the beta distribution.

Next, we propagate to  $\theta_t$  using  $(M_t^{k(i)}, \theta_{t-1}^{k(i)})$ , and for this we need the density  $p(\theta_t | \theta_{t-1}, M_t, M^{(t-1)}, Y^{(t)})$ ,

$$p(\theta_t | \theta_{t-1}, M_t, Y^{(t)}, M^{(t-1)}) \propto p(Y_t | M_t, \theta_t) p(\theta_t | M_t, \theta_{t-1}, M^{(t-1)}, Y^{(t-1)})$$

which can be written as

$$p(\theta_t | \theta_{t-1}, M_t, Y^{(t)}, M^{(t-1)}) \propto e^{-\theta_t} \left( \frac{\theta_{t-1}}{\gamma} - \theta_t \right)^{(1-\gamma)a_{t-1}-1} \theta^{\gamma a_{t-1} + (Y_t - M_t) - 1}. \quad (21)$$

The above is proportional to a scaled hypergeometric beta density as discussed by Gordy (1998). We can sample from (21) using Metropolis-Hastings or some form of rejection sampling.

Alternatively, in propagating  $\theta_t$ , if generating from the hypergeometric beta density in (21) is not computationally efficient, we can use a sequential importance sampling type step as suggested in Aktekin et. al (2018). More specifically,  $\theta_t$  can be (blind) propagated via

$$p(\theta_t | \theta_{t-1}, M^{(t-1)}, Y^{(t-1)}),$$

which is a scaled Beta distribution,  $Beta(\gamma a_{t-1}, (1 - \gamma) a_{t-1}, \theta_{t-1}/\gamma)$ . The particles are then resampled with weights

$$p(Y_t | \theta_t, M_t),$$

which is a shifted-*Poisson*( $\theta_t$ ), with  $Y_t \geq M_t$ . Finally, we can update  $\gamma$  offline or alternatively using a discrete prior with the marginal likelihood given by (19). Algorithm 2 provides a schematic representation of this particle filter algorithm.

### Collapsed particle filtering.

An alternative to particle filtering that avoids the evaluation of CHF is to integrate out  $\theta_t$  and using the predictive distribution of  $\epsilon_t = Y_t - M_t$  given  $M^{(t-1)}, Y^{(t-1)}$  for resampling. we refer to this approach as "collapsed particle

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**Algorithm 2** Particle filter algorithm
 

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- 1: Let  $\{(\theta_{t-1}, M_{t-1}, \alpha)^i\}_{i=1}^N$  be the particle set at time  $t - 1$ .
  - 2: Draw  $M_t^i \sim \text{Binomial}(Y_{t-1}, \alpha^i)$ .
  - 3: Compute resampling weights from Equation (18)
  - 4: Resampling particles  $\{(\theta_{t-1}, M_t)^{k(i)}\}_{i=1}^N$  with weights from 2.
  - 5: Update  $(a_t, b_t)$ :  $a_t^i = \gamma a_{t-1}^{k(i)} + (Y_t - M_t^{k(i)})$  and  $b_t^i = \gamma b_{t-1}^{k(i)} + 1$ .
  - 6: Draw thinning:  $\alpha | M^t, Y^t \sim \text{Beta}(S_{1t}^i, S_{2t}^i)$ , where  $S_{1t}^i = S_{1,t-1}^{k(i)} + M_t^{k(i)}$  and  $S_{2t}^i = S_{2,t-1}^{k(i)} + (Y_{t-1} - M_t^{k(i)})$ , where  $S_{10} = a_\alpha$  and  $S_{20} = b_\alpha$ .
  - 7: Propagate  $\theta_t$  from  $\theta_t \sim \text{Beta}(\gamma a_{t-1}, (1 - \gamma) a_{t-1}, \theta_{t-1} / \gamma)$ , a scaled Beta distribution.
  - 8: Resample with weights  $\rho(Y_t | \theta_t, M_t)$ , a shifted *Poisson*( $\theta_t$ ), with  $Y_t \geq M_t$ .
- 

filtering".

Assume that in the time period  $t - 1$  given  $Y^{(t-1)}$ , we have particles of  $(M_{t-1}^i, \alpha^i)$   $i = 1, \dots, N$ . Given these particles, we can obtain particles from  $M_t^i$  using the binomial distribution  $\text{Bin}(Y_{t-1}, \alpha^i)$  for  $i = 1, \dots, N$ . These can be considered as "prior" particles for  $M_t$  and at time  $t$  we need to update these given  $Y_t$ . In so doing, We can resample  $M_t$ 's using the predictive distribution

$$p(\epsilon_t | \gamma, M^{(t-1)}, Y^{(t-1)}) = \frac{\Gamma(\gamma a_{t-1} + \epsilon_t)}{\Gamma(\epsilon_t + 1) \Gamma(\gamma a_{t-1})} \left(1 - \frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{\epsilon_t} \left(\frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{\gamma a_{t-1}}.$$

In other words, we use the above predictive distribution as the resampling weight to obtain the posterior particles  $M_t^{k(i)}$  and update  $\alpha$  using the particles  $M_t^{k(i)}$  in the beta distribution. Note that once we have the posterior particles of we can update the sufficient statistics in the posterior distribution of  $(\theta_t | \gamma, M_t, M^{(t-1)}, Y^{(t-1)}) \sim \text{Gamma}(a_t, b_t)$  where  $a_t = \gamma a_{t-1} + (Y_t - M_t^{k(i)})$  and  $b_t = \gamma b_{t-1} + 1$ . In Algorithm 3, we present a schematic representation of the collapsed particle filtering algorithm.

## 4 | DYNAMIC INAR(P) PROCESSES

Consider the p-th order INAR setup presented in Du and Li (1991) as

$$Y_t = \alpha_1 \circ Y_{t-1} + \alpha_2 \circ Y_{t-2} + \dots + \alpha_p \circ Y_{t-p} + \epsilon_t,$$

where

$$\alpha_j \circ Y_{t-j} = \sum_{i=1}^{Y_{t-j}} B_{ij t}$$

and  $B_{ij t}$ 's are Bernoulli random variables with success probability  $\alpha_j$ ,  $j = 1, 2$ . For stationarity of the INAR(p) model we need  $0 < \sum_{j=1}^p \alpha_j < 1$ . Given  $Y_{t-j}$ ,  $\alpha_j \circ Y_{t-j}$  has a binomial distribution with parameters  $\alpha_j$  and  $Y_{t-j}$ .

In the above setup it is assumed that  $\{\epsilon_t\}$  is a sequence of independent Poisson random variables with parameter  $\theta_t$  and  $B_{ij t}$ 's are independent of  $\epsilon_t$ 's. It is important to note that thinning is performed at each time point  $t$  and is

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**Algorithm 3** Collapsed particle filter algorithm
 

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- 1: Let  $\{(M_{t-1}, \alpha)^i\}_{i=1}^N$  be the particle set at time  $t - 1$ .
- 2: Draw  $M_t^i \sim \text{Binomial}(Y_{t-1}, \alpha^i)$ .
- 3: Compute resampling weights from equation (12):

$$\omega^i = \frac{\Gamma(\gamma a_{t-1} + e_t^i)}{\Gamma(e_t^i + 1) \Gamma(\gamma a_{t-1})} \left(1 - \frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{e_t^i} \left(\frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{\gamma a_{t-1}},$$

where  $e_t^i = Y_t - M_t^i$ .

- 4:  $M_t^{k(i)}$  are the resample draws.
- 5: Update sufficient statistics and sample thinning probability  $\alpha^i$

$$\begin{aligned} S_{1t}^i &= S_{1,t-1}^{k(i)} + M_t^{k(i)} \\ S_{2t}^i &= S_{2,t-1}^{k(i)} + (Y_{t-1} - M_t^{k(i)}) \\ \alpha^i &\sim \text{Beta}(S_{1t}^i, S_{2t}^i), \end{aligned}$$

where  $S_{10} = a_\alpha$  and  $S_{20} = b_\alpha$ .

- 6: Update  $a_t$  and  $b_t$  and sample  $\theta_t$  from  $(\theta_t | \gamma, M^{(t)}, Y^{(t)})$

$$\begin{aligned} a_t^i &= \gamma a_{t-1}^{k(i)} + (Y_t - M_t^{k(i)}) \\ b_t^i &= \gamma b_{t-1}^{k(i)} + 1 \\ \theta_t^i &\sim \text{Gamma}(a_t^i, b_t^i). \end{aligned}$$


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independent of thinning at other periods. We define latent variables  $M_{jt} = \alpha_j \circ Y_{t-j}$  where

$$M_{jt}|Y_{t-j}, \alpha_j \sim \text{Bin}(Y_{t-j}, \alpha_j)$$

and  $Y_t|M_{1t}, \dots, M_{pt}, \theta_t$  is a shifted Poisson, that is,

$$\rho(Y_t|M_{1t}, M_{2t}, \dots, M_{pt}, \theta_t) = \frac{e^{-\theta_t} \theta_t^{Y_t - \sum_{j=1}^p M_{jt}}}{(Y_t - \sum_{j=1}^p M_{jt})!} \mathcal{I}(Y_t \geq \sum_{j=1}^p M_{jt}).$$

Note that based on the above  $0 \leq M_{jt} \leq \min(Y_t - \sum_{i \neq j} M_{it}, Y_{t-j})$  and if  $(Y_t - \sum_{i \neq j} M_{it}) = 0$  or  $Y_{t-j} = 0$  then  $M_{jt} = 0$  in the model. Also, if  $Y_t = 0 \Rightarrow M_{1t} = M_{2t} = \dots = M_{pt} = 0$ .

Using the Markov evolution of  $\theta_t$  given by (6), we can develop Bayesian inference for the dynamic INAR(p) model. In this case, we define  $M_t = (M_{1t}, \dots, M_{pt})$  as a vector. It can be shown that the filtering distribution of  $\theta_t$  given by (9) will have parameters  $a_t = \gamma a_{t-1} + (Y_t - \sum_{j=1}^p M_{jt})$  and  $b_t = \gamma b_{t-1} + 1$ . Also, for stationarity of the model we assume a Dirichlet prior for vector  $\alpha = (\alpha_1, \dots, \alpha_p)$  as

$$\rho(\alpha) \propto \prod_{j=1}^{p+1} \alpha_j^{a_{\alpha_j} - 1}, \quad (22)$$

where  $a_{\alpha_j} > 0$ , for  $j = 1, \dots, p+1$ , and  $\alpha_{p+1} = 1 - \sum_{j=1}^p \alpha_j$ .

### Gibbs sampler.

As in Section 3.1, given the above structure, we can obtain the full conditionals of all the quantities in the dynamic INAR(p) model. For the maturation terms,  $M_{jt}$ 's we can show that

$$\rho(M_{jt}|M^{(-jt)}, \theta^{(T)}, Y^{(T)}) \propto \left\{ \prod_{s=t+1}^T \rho(\theta_s|\theta_{s-1}, M^{(s-1)}, Y^{(s-1)}) \right\} \rho(M_{jt}|Y_{t-1}) \rho(Y_t|M_t, \theta_t)$$

where  $M_{jt} \in \{0, 1, \dots, \min(Y_t - \sum_{i \neq j} M_{it}, Y_{t-j})\}$  for  $t = (p+1), \dots, T-1$  and  $j = 1, \dots, p$ . For  $t = T$  and  $j = 1, \dots, p$

$$\rho(M_{jT}|M^{(-jT)}, \theta^{(T)}, Y^{(T)}) \propto \rho(M_{jT}|Y_{T-j}) \rho(Y_T|M_T, \theta_T).$$

For components of  $\alpha$ , using (20), we have conditional priors of  $\alpha_j$ 's as scaled beta densities

$$\rho(\alpha_j|\alpha^{(-j)}) \propto \alpha_j^{a_{\alpha_j} - 1} (1 - \sum_{j=1}^p \alpha_j)^{a_{\alpha_{p+1}} - 1},$$

where  $\alpha_j \in (0, 1 - \sum_{i \neq j} \alpha_i)$ . It can be shown that the posterior full conditionals of  $\alpha_j$ 's can be obtained as scaled beta densities as

$$\alpha_j|\alpha^{(-j)}, M^{(T)}, Y^{(T)} \sim \text{Beta}(a_{\alpha_j} + \sum_{t=p+1}^T M_{jt}, a_{\alpha_{p+1}} + \sum_{t=p+1}^T (Y_{t-j} - M_{jt})) \mathcal{I}(\alpha_j < 1 - \sum_{i \neq j} \alpha_i). \quad (23)$$

Learning of  $\theta^{(T)}$  and the discount parameter  $\gamma$  follow along the same lines as discussed in Section 3.1.

## Particle filtering.

The sequential Monte Carlo approach of Section 3.2 can be modified for the INAR(p) extension. Specifically, in Algorithm 2, in step 1, we need to draw  $M_{jt}^i \sim \text{Binomial}(Y_{t-j}, \alpha_j^i)$  for  $j = 1, 2, \dots, p$ . The resampling weights of equation (18) are now computed by replacing  $(Y_t - M_t)$  with  $(Y_t - \sum_{j=1}^p M_{jt}^i)$ . This also applies to the resampling particles of step 3 and to the updates of  $(a_t, b_t)$  in Step 4. Propagation of  $\theta_t$  follows the step 6 of the Algorithm 2, and finally  $\theta_t$ 's can be resampled using weights based on a shifted Poisson with  $Y_t \geq \sum_{j=1}^p M_{jt}^i$ . However, since there are no sufficient statistics for  $(\alpha_1, \dots, \alpha_p)$ , in step 5, a mixture-based approach as in Liu and West (2001) can be used to draw the thinning probabilities.

Alternatively, by not enforcing stationarity, independent beta priors can be specified for  $(\alpha_1, \dots, \alpha_p)$  and sufficient statistics can be updated in a similar manner as in step 5 of Algorithm 3. In what follows, we present a collapsed PF by assuming that  $\alpha_j$ 's have independent beta priors with parameters  $a_{\alpha_j}, b_{\alpha_j}$  for  $j = 1, \dots, p$ .

Without loss of generality, we consider the case of  $p = 2$  where we have the following collapsed PF algorithm:

Let  $\{(M_{1,t-1}, M_{2,t-1}, \alpha_1, \alpha_2)^j\}_{j=1}^N$  be the particle set at time  $t - 1$ .

1. Draw  $M_{1t}^i \sim \text{Binomial}(Y_{t-1}, \alpha_1^i)$  and  $M_{2t}^i \sim \text{Binomial}(Y_{t-2}, \alpha_2^i)$
2. Compute resampling weights from equation (10):

$$\omega^j = \frac{\Gamma(\gamma a_{t-1} + e_t^j)}{\Gamma(e_t^j + 1) \Gamma(\gamma a_{t-1})} \left(1 - \frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{e_t^j} \left(\frac{\gamma b_{t-1}}{\gamma b_{t-1} + 1}\right)^{\gamma a_{t-1}},$$

where  $e_t^j = Y_t - M_{1t}^i - M_{2t}^i$ . We use the above predictive distribution as the resampling weight to obtain the posterior particles  $M_{1t}^{k(i)}, M_{2t}^{k(i)}$  and update  $\alpha_1, \alpha_2$  using these particles in the beta distribution

3. Update sufficient statistics and sample thinning probability  $\alpha^i$

$$\begin{aligned} S_{11t}^i &= S_{11,t-1}^{k(i)} + M_{1t}^{k(i)} \\ S_{12t}^i &= S_{12,t-1}^{k(i)} + (Y_{t-1} - M_{1t}^{k(i)}) \\ \alpha_1^i &\sim \text{Beta}(S_{11t}^i, S_{12t}^i), \end{aligned}$$

and similarly

$$\begin{aligned} S_{21t}^i &= S_{21,t-1}^{k(i)} + M_{2t}^{k(i)} \\ S_{22t}^i &= S_{22,t-1}^{k(i)} + (Y_{t-2} - M_{2t}^{k(i)}) \\ \alpha_2^i &\sim \text{Beta}(S_{21t}^i, S_{22t}^i), \end{aligned}$$

where  $S_{110} = a_{\alpha_1}, S_{120} = b_{\alpha_1}$  and  $S_{210} = a_{\alpha_2}, S_{220} = b_{\alpha_2}$ .

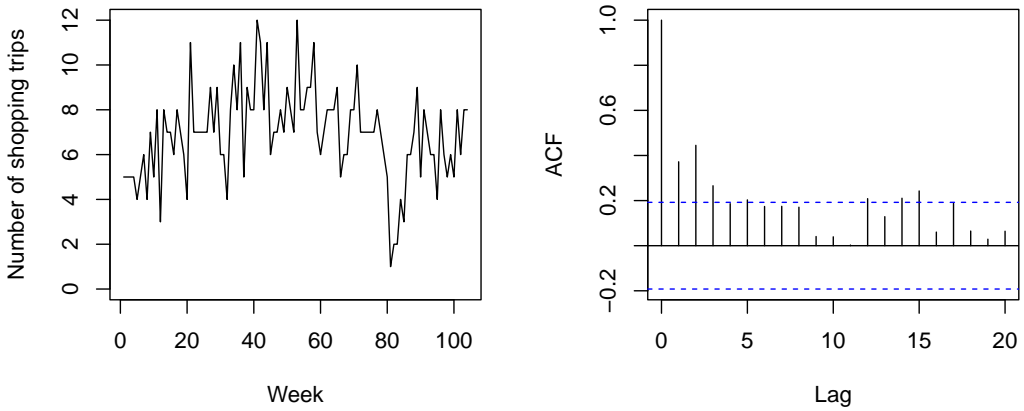
4. Update  $a_t$  and  $b_t$  and sample  $\theta_t$  from  $(\theta_t | \gamma, M^{(t)}, Y^{(t)})$

$$\begin{aligned} a_t^i &= \gamma a_{t-1}^{k(i)} + (Y_t - M_{1t}^{k(i)} - M_{2t}^{k(i)}) \\ b_t^i &= \gamma b_{t-1}^{k(i)} + 1 \\ \theta_t^i &\sim \text{Gamma}(a_t^i, b_t^i). \end{aligned}$$

## 5 | NUMERICAL ILLUSTRATIONS

To illustrate the implementation of the Gibbs sampler and particle filtering algorithms, we consider actual data on the number of weekly shopping trips for a household to the supermarket over 104 weeks. The data is a subset of a large set used in Kim (2013).

In Figure 1 we show the weekly time series and its sample autocorrelation function (ACF). We note that the time series exhibits moderate-level autocorrelations at lags 1 and 2.



**FIGURE 1** Number of weekly shopping trips time series and sample ACF.

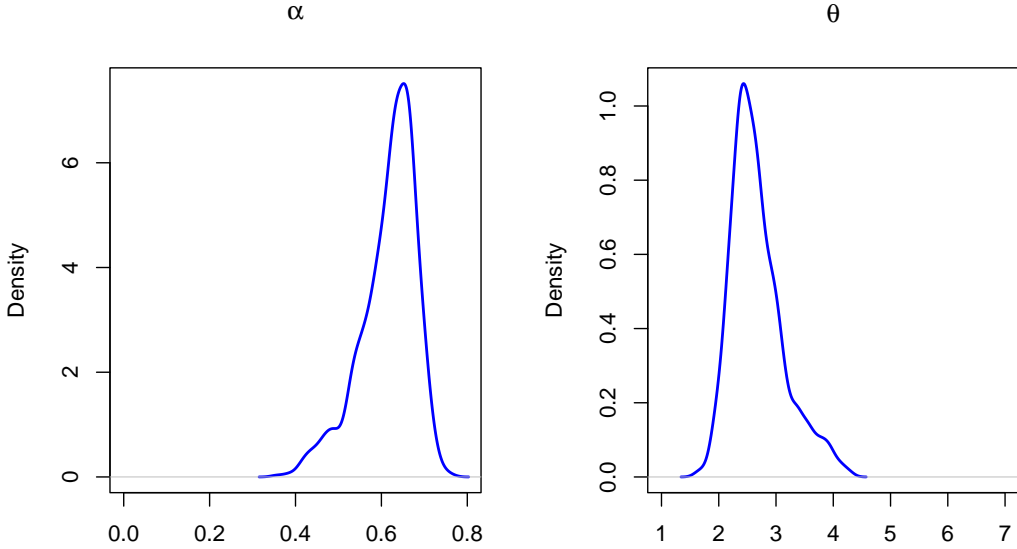
Overall sample mean and variance for 104 weeks are 6.92 and 4.26, respectively. However, as can be seen from Table 1, there is evidence suggesting that mean is not constant over the 104 week period.

Weeks	Sample mean	Sample variance
1-26	6.19	2.88
27-52	8.04	3.72
53-78	7.81	2.40
79-104	5.65	4.24

**TABLE 1** Mean and variance of weekly shopping trips over different time periods.

### Static INAR(1).

Next, we consider the Bayesian analysis of the static INAR(1) model where  $\theta_t = \theta$  for all  $t$ . We will use the static model as a reference in our analysis. The posterior distributions of  $\alpha$  and  $\theta$  from the static INAR(1) model are shown in Figure 2. We can obtain the posterior means of  $\alpha$  and  $\theta$  as 0.622 and 2.565, respectively. As previously mentioned, the static



**FIGURE 2** Marginal posterior distributions for  $\alpha$  and  $\theta$  based on the static INAR(1) model.

model arises as a special case of the dynamic INAR(1) model as the value of the discount parameter  $\gamma$  approaches to 1. In other words, these results can be replicated by using a Gibbs sampler or the PF algorithm when we set  $\gamma$  to a large value such as 0.999.

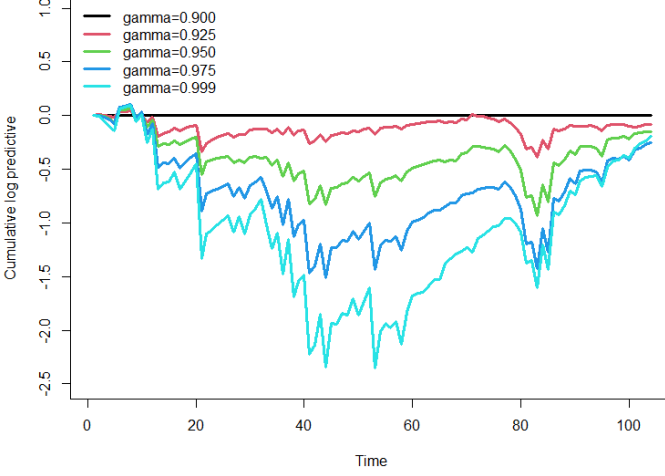
### Dynamic INAR(1).

The use of the Gibbs sampler is not computationally efficient for sequential updating, as it requires the sampler to be rerun for every data point observed. Therefore, in what follows, we will focus on the collapsed particle filtering results. The choice of the discount parameter  $\gamma$  plays an important role in the Bayesian analysis of the dynamic INAR(1) model. As discussed in Section 3.2, we can learn about  $\gamma$  using the marginal likelihood in (19) sequentially using the Monte Carlo (MC) average of the updated samples of  $M^{(t)}$  at each time point  $t$ . Alternatively, one can use the marginal likelihood (19) to specify a fixed value of the discount factor, which is the most supported by the data as in West and Harrison (1997). In other words, using the posterior samples of  $M^{(t)}$ , we can obtain a MC average of (19) for fixed values of  $\gamma$  and find the  $\gamma$  that maximizes the MC average. This can be done sequentially over time, in the sense of Dawid (1984), to see which value of  $\gamma$  is supported dynamically by the data.

It is important to note that  $\gamma$  is a tuning parameter in dynamic models. It represents the change in uncertainty about  $\theta$  values neighboring each other. Based on the Markovian model presented in Section 2.2, we can show that the percent increase in uncertainty (in terms of variance) from time  $(t-1)$  to  $t$  is given by  $(1-\gamma)/\gamma$  for all  $t$ . As noted by West and Harrison (1997, page 51), it is common practice to select  $\gamma \in (0.8, 1)$ .

We consider the values of  $\gamma = 0.900, 0.925, 0.950, 0.975$  and  $0.999$  and evaluate the MC average of the marginal likelihood (19) based on the particles of  $M^{(t)}$  at each time point  $t = 2, \dots, T$ . Based on the results of the collapsed

particle filtering, in Figure 3, we illustrate the log of cumulative marginal likelihoods for different values  $\gamma$  relative to  $\gamma = 0.9$ . Figure 3 suggests that the static INAR(1) model (which is represented by  $\gamma = 0.999$ ) is not supported by the data. Based on the figure, we will use  $\gamma = 0.9$  in our analysis of the dynamic INAR(1) model.



**FIGURE 3** Cumulative marginal likelihood for different values  $\gamma$  from INAR(1), relative to  $\gamma = 0.9$ .

The posterior distribution of  $\alpha$  obtained using the collapsed particle filter is shown in Figure 4, suggesting that the median values of the probability of thinning stabilize around the values of 0.55 – 0.60. The posterior (filtering) distributions of the  $\theta_t$ 's, when  $\gamma = 0.9$  are shown in Figure 5. As expected, the filtering distributions of  $\theta_t$ 's in this case exhibit more variability over time compared to their counterparts from models with higher values of  $\gamma$ , which are not shown here.

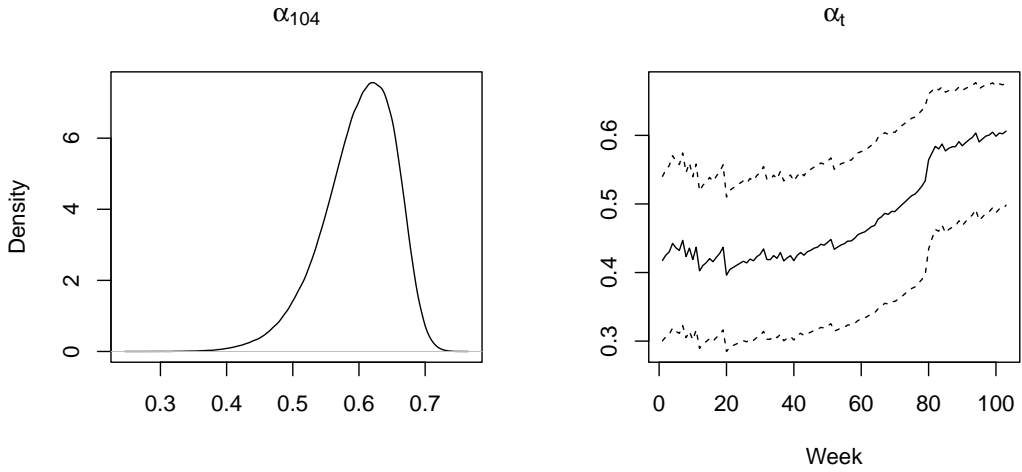
In Figure 6, we present the plot of  $E[M_t + \theta_t | Y^t]$  versus actual values of  $Y_t$ 's for  $t = 2, \dots, T$ . The figure illustrates the fit of the dynamic INAR(1) model to the actual values of number of shopping trips. We can look at the predictive means of  $Y_t$ 's at time  $t - 1$  to assess the forecast performance of the model. Specifically, we can use

$$E[Y_t | Y^{(t-1)}] = E[\alpha | Y^{(t-1)}] Y_{t-1} + E[\theta_t | Y^{(t-1)}]$$

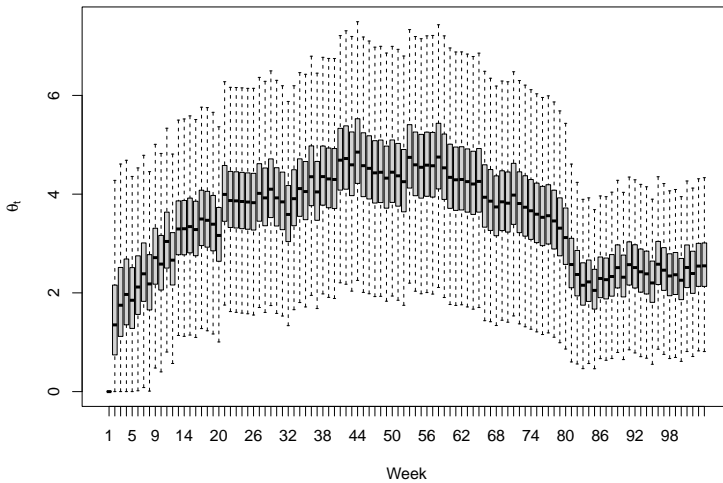
which can be obtained as a MC average for time periods  $t = 2, \dots, T$ .

We can alternatively use the medians of the predictive distribution  $Y_t$ 's at time  $t - 1$  to assess the forecast performance. To obtain the median of  $Y_t$  at time  $(t - 1)$  we need the draws from the distribution of  $Y_t$ . Given the  $N$  particles from the distributions  $p(\alpha | Y^{(t-1)})$ ,  $p(\theta_{t-1} | Y^{(t-1)})$ , for  $i = 1, \dots, N$  we can draw from the predictive distribution of  $Y_t$  using the following.

1. Draw  $M_t^i \sim \text{Binom}(\alpha^i, Y_{t-1})$  and draw  $\theta_t^i \sim \text{Gamma}(\gamma a_{t-1}, \gamma b_{t-1})$
2. Draw  $\epsilon_t^i \sim \text{Pois}(\theta_t^i)$
3. Obtain  $Y_t^i = M_t^i + \epsilon_t^i$ .



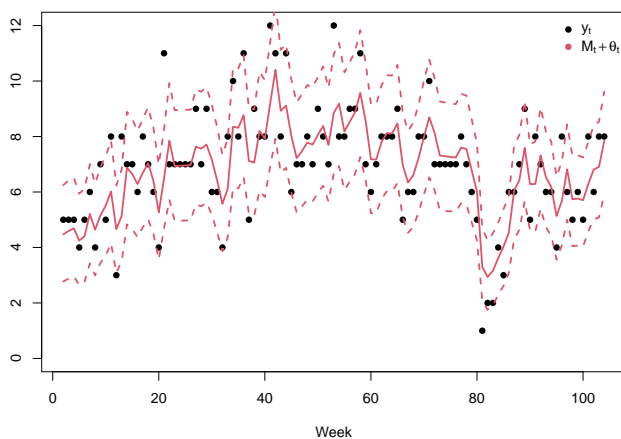
**FIGURE 4** Marginal posterior distribution of  $\alpha$  at time  $t = 104$  (left) and 5th, 50th and 95th percentiles of  $\rho(\alpha|Y^{(t)})$ , for  $t = 1, \dots, 104$  (right), when  $\gamma = 0.9$ .



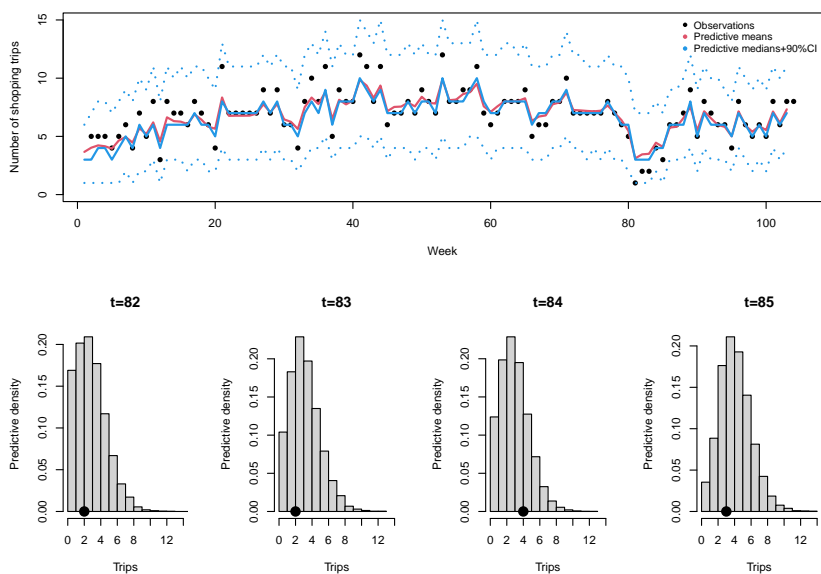
**FIGURE 5** Filtering distributions of  $\theta_t$ 's in INAR(1) model when  $\gamma = 0.9$ .

We present the plot of actual versus one-step ahead forecasts using predictive means (shown in red) and predictive medians (in blue) in Figure 7 of  $Y_t$ 's at time  $t - 1$  along with 90 per cent Bayesian credibility intervals. At the bottom portion of the figure the posterior predictive densities  $\rho(Y_t|Y^{(t-1)})$  are shown for  $t = 82, \dots, 85$  with the observed values of  $Y_t$  are displayed as a  $\bullet$ . It is important to note that these time points correspond to predictive distributions

which are more skewed, but in spite of that the predictive performance is quite satisfactory.



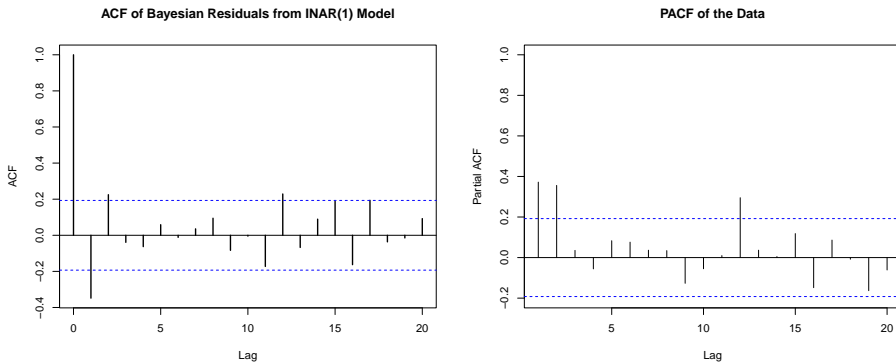
**FIGURE 6** Actual number of trips versus  $E(M_t + \theta_t | Y^t)$  in INAR(1) model.



**FIGURE 7** Top row: Observed number of shopping trips versus predicted means, predictive medians and predictive 90% credibility intervals. Bottom row: Predictive densities,  $\rho(Y_t | Y^{(t-1)})$ , for  $t = 82, \dots, 85$ .

## 5.1 | Comparing INAR(1) and INAR(2)

In Figure 8, we present the ACF of the residuals from the INAR(1) model based on predictive means. The partial autocorrelation function (PACF) of the data is also shown on the figure. The ACF of the residuals still displays autocorrelation at lag 1. Since the PACF of the original data exhibits autocorrelations both at lags 1 and 2, an INAR(2) process may be more appropriate in this case. This will be considered next.



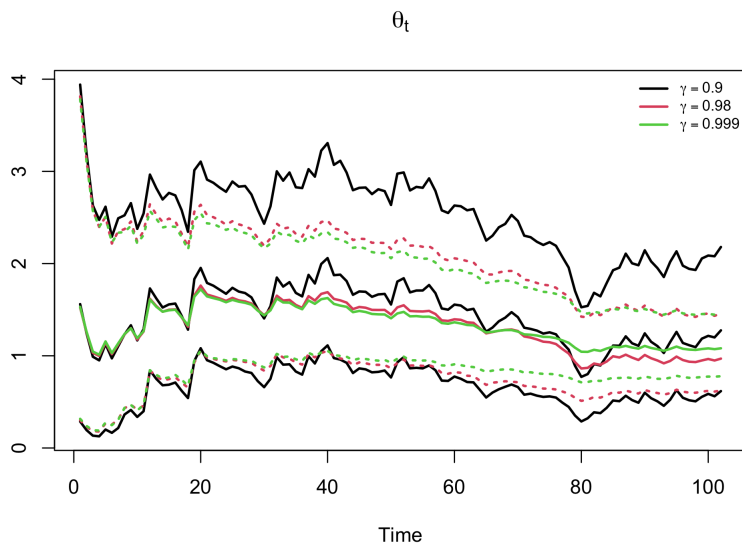
**FIGURE 8** ACF of residuals from the INAR(1) model and the PACF of the data.

We compare the results of dynamic INAR(1) with  $\gamma = 0.9$  with the implementations of dynamic INAR(2) for various values of  $\gamma$ . The INAR(2) model analysis is developed using the collapsed PF results presented in Section 4. Evaluation of the marginal likelihood sequentially for different values of  $\gamma \in (0.900, 0.999)$ , under the INAR(2) model, suggests  $\gamma = 0.98$  is favored. Figure 9 shows the quantiles of  $\theta_t$  for  $\gamma = 0.9, 0.98, 0.999$ . As expected  $\theta_t$  under  $\gamma = 0.98$  exhibits a smoother behavior over time compared to the case of  $\gamma = 0.9$ . It is important to note that the static INAR(2) model which is corresponding to  $\gamma = 0.999$  in Figure 9 (shown in green) can not adapt to the drop in the data in week 81. Figure 10 shows the sequential Bayes factor for the INAR(2) model for  $\gamma \in \{0.9, 0.98, 0.999\}$ . As it can be observed,  $\gamma = 0.98$  is better than  $\gamma = 0.999$ , which is better than  $\gamma = 0.9$ . The evidence in favor of the dynamic INAR(2) over the static INAR(2) becomes more pronounced after week 81 where the data exhibits a drop.

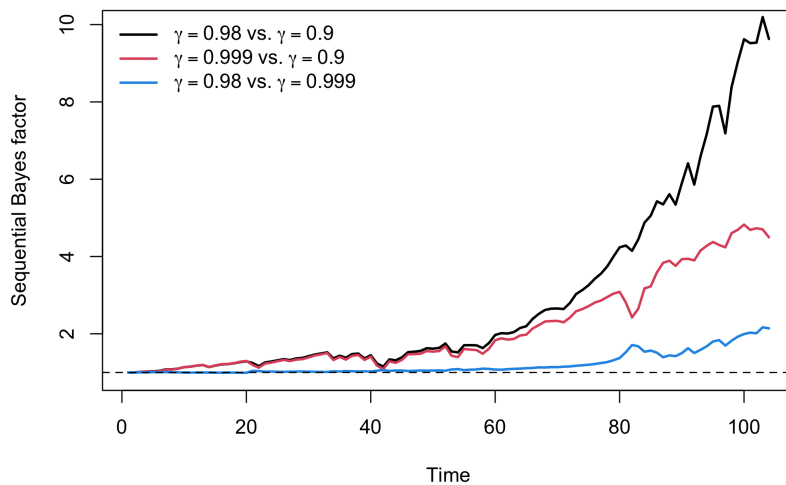
Figure 11 presents the sequential posterior quantiles for  $\alpha_1$  and  $\alpha_2$  in INAR(2). As can be seen, their time-varying behaviors are quite similar, with only minor variations at the end of the sample size. In any event,  $\alpha_1$  seems to be around 0.3, while  $\alpha_2$  settles around 0.6.

Finally, Figure 12 presents the cumulative logarithmic Bayes factor for the top INAR(2) model relative to the INAR(1) model. As can be argued, after 20 observations, the INAR(2) fit becomes overly better than the INAR(1). The ACF of the residuals are presented in Figure 13. Unlike the INAR(1) residuals, the residuals from the dynamic INAR(2) model exhibit no considerable autocorrelation at any of the lags, which also confirms the appropriateness of the dynamic INAR(2) model.

In conclusion, the Bayesian viewpoint provides quantification of uncertainty via probability. As a result, all inferences, forecasts, and model comparisons are based on probabilistic statements which are derived using calculus of probability. This is attractive from an applied perspective, since additional insights can be obtained from Bayesian analysis using the posterior and/or predictive distributions. For example, in the dynamic INAR(2) model, a hypothesis such as  $\alpha_1 < \alpha_2$  can be resolved simply using the joint posterior distribution of  $\alpha_1$  and  $\alpha_2$ . Similarly, in our use of collapsed particle filtering for the dynamic INAR(2) model with independent beta priors for  $\alpha_1$  and  $\alpha_2$ , we are able to



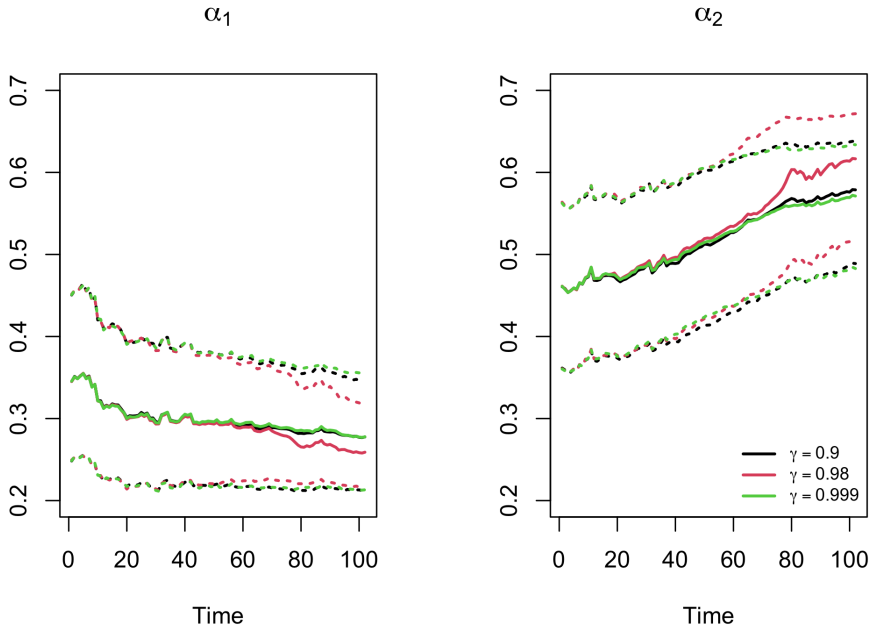
**FIGURE 9** Sequential posterior quantiles (5%, 50% and 95%) for the state space  $\theta_t$  at various values of  $\gamma$ .



**FIGURE 10** Sequential Bayes factor for the INAR(2) model for various values of  $\gamma$ .

assess if  $\alpha_1 + \alpha_2 < 1$  which is the condition for stationarity of the static INAR(2).

Our proposed sequential Bayesian predictive framework—based on full predictive distributions—combined with

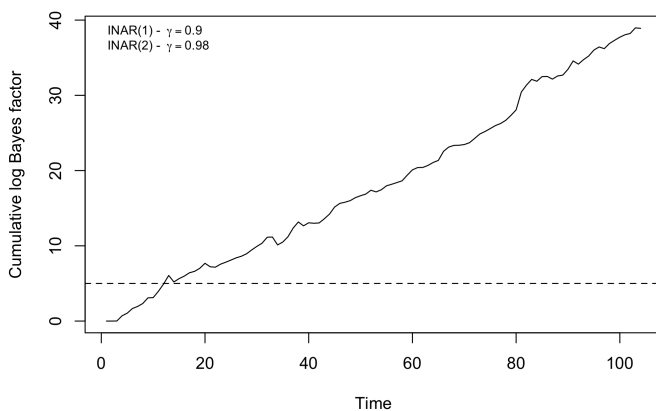


**FIGURE 11** Sequential posterior quantiles (5%, 50% and 95%) for both thinning parameters  $\alpha_1$  and  $\alpha_2$  at various values of  $\gamma$ .

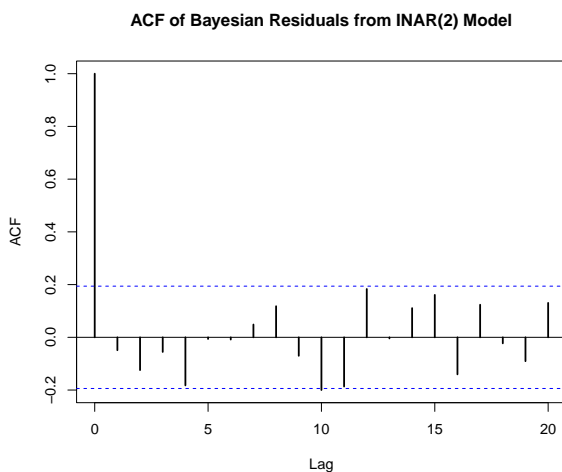
careful model specification and selection, provides a natural approach to achieving calibration in the sense of Philip Dawid (1982). Model comparison in our framework is conducted probabilistically via Bayes factors, which decompose into cumulative predictive likelihoods and are therefore consistent with the prequential perspective.

In addition, certain frequentist tools, such as the sample autocorrelation and partial autocorrelation functions, can be employed for model identification (see Figures 1 and 8). Further, examination of the sample autocorrelations of Bayesian residuals (Figures 8 and 13) provides a useful diagnostic for model adequacy.

The use of frequentist tools for model validation was advocated by George Box (1980), who emphasized Bayesian model-based inference while employing frequentist ideas for model criticism. Little (2006, 2011) extended this perspective by emphasizing the frequentist performance of Bayesian procedures, referring to this approach as "calibrated Bayes". It is important to note that Little's notion of calibration is based on repeated sampling properties, which differs fundamentally from Dawid's (1982) concept of calibration, grounded in sequential predictive performance rather than hypothetical repeated sampling.



**FIGURE 12** Comparing the best INAR(2) model, where  $\gamma = 0.98$ , to the best INAR(1) model, where  $\gamma = 0.9$ , in terms of cumulative log predictive densities. The dashed line represents the threshold for *very strong evidence against the INAR(1) model*, based on the suggestions by Kass and Raftery (1995, pg 777).



**FIGURE 13** ACF of residuals from the INAR(2) model

## 6 | CONCLUDING REMARKS

We introduced a new class of dynamic integer autoregressive (INAR) models for count data. The proposed class of models include static INAR models as well as some of the parameter driven Bayesian time series models for counts as special cases. Bayesian analysis of the proposed model was developed using a Gibbs sampler and particle filtering algorithms were introduced for sequential analysis of the model. Extension of the dynamic model to high-order INAR processes was discussed. Using a real life time series of counts, the dynamic INAR(2) model was shown to perform better than its static counter part as well as the dynamic INAR(1) process. Alternative dynamic INAR processes and the incorporation of covariates into the models were discussed as possible extensions that are currently under investigation. Our future work also involves development of dynamic multivariate INAR processes.

It is important to note that the proposed dynamic INAR(p) models and the presented Bayesian MCMC and sequential MC approaches are novel and differ from the previously considered state-dependent Bayesian INAR models in several respects. First of all, the proposed class of Bayesian models' inherent conditional conjugacy allows for using the Gibbs sampler without any Metropolis steps. Secondly, the proposed PF algorithms exploit this conjugacy and the availability of sufficient statistics to update static parameters and provide efficient sequential inference and forecasting. Such a sequential analysis cannot be performed efficiently by previously introduced Bayesian models that relied on MCMC methods.

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