An Introduction to Bayes' Theorem

Belief Revision Under Uncertainty

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A systematic approach to making decisions under uncertainty

Outline

The Motivating Problem

Scenario: You are sitting on your patio and spot a wisp of smoke in the sky.

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- You cannot see flames or determine the source
- **Question:** Is there a fire in the neighbourhood?

Two Hypotheses

- H: **Fire** is occurring
- $\neg H$: **No fire** is occurring

Belief Revision

Start with prior beliefs

$\downarrow$

Observe data

$\downarrow$

Update beliefs

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Prior Probabilities

Priors encode beliefs *before* any data is observed.

Conjectures Supporting $P(H) = 0.10$

- **Rarity of fires:** You've lived in many neighbourhoods and never witnessed one
- **Summer season:** Neighbours are frequently barbecuing — a benign smoke source
- **Low bushfire risk:** Neighbourhood is not near heavily wooded areas
- **New construction:** Houses < 5 years old $\Rightarrow$ low risk of electrical faults

$P(H) = 0.10$ *(Fire – 10% chance)*

$P(\neg H) = 1 - P(H) = 0.90$ *(No fire – 90% chance)*

Likelihoods

Likelihood $P(D \mid H)$ — probability of observing data D (smoke) given hypothesis H.

$P(D \mid H) = 0.90$

Conjecture: If a fire is burning, it almost always produces visible smoke.

Allows for a rare 10% chance of fire without visible smoke.

$P(D \mid \neg H) = 0.15$

Conjecture: In summer, barbecues are a common non-fire source of smoke — active for a few months of the year.

A 15% chance of smoke with no fire.

Hypothesis	**Smoke (D)**	**No Smoke ($\neg D$)**
Fire (H)	0.90	0.10
No Fire ($\neg H$)	0.15	0.85

Each row sums to 1 (conditional on the hypothesis being true).

Joint Probabilities

Joint probability $P(H \cap D)$ — probability that *both* H and D are true.

Product Rule:

$$P(H \cap D) = P(D \mid H) \cdot P(H) = 0.90 \times 0.10 = \mathbf{0.09}$$

Hypothesis	**Smoke (D)**	**No Smoke ($\neg D$)**	**Total**
Fire (H)	0.09	0.01	0.10
No Fire ($\neg H$)	0.14	0.76	0.90
Total	0.23	0.77	1.00

Marginalisation (rows):
Summing across columns recovers the prior: $\sum_D P(H \cap D) = P(H)$

Law of Total Probability (columns):
$P(D) = P(D|H)P(H) + P(D|\neg H)P(\neg H) = 0.23$

Bayes' Theorem

Once smoke D is actually observed, all scenarios with $\neg D$ are eliminated.

We update our beliefs using the **rule of conditional probability**:

Bayes' Theorem

$$\underbrace{P(H \mid D)}_{\text{Posterior}} = \frac{\overbrace{P(D \mid H)}^{\text{Likelihood}} \times \overbrace{P(H)}^{\text{Prior}}}{\underbrace{P(D)}_{\text{Marginal}}}$$

Substituting the values:

$$P(H \mid D) = \frac{0.09}{0.23} \approx \mathbf{0.39}$$

- **Prior:** $P(H) = 0.10 \quad \longrightarrow \quad$ **Posterior:** $P(H \mid D) \approx 0.39$
- Seeing smoke causes a nearly 4-fold increase in the probability of fire.

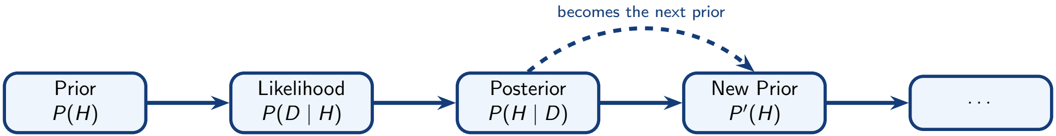
Summary: Conjectures Behind Each Probability

Probability	Value	Key Conjectures
$P(H)$	0.10	Fires are rare; new builds; no nearby forest; summer BBQ season
$P(\neg H)$	0.90	Complement of $P(H)$
$P(D \mid H)$	0.90	Fire almost always produces visible smoke
$P(D \mid \neg H)$	0.15	Summer BBQs are a common benign smoke source
$P(D)$	0.23	Derived via law of total probability
$\mathbf{P(H \mid D)}$	**0.39**	**Posterior after observing smoke – nearly 4$\times$ prior**

Key Takeaway

Qualitative background knowledge (season, construction age, proximity to forests) is formalised into probabilities, which then drive a principled Bayesian update.

The Iterative Power of Bayesian Updating



- The posterior becomes the prior for the **next** update.
- Beliefs can be **continuously refined** as new evidence arrives.
- E.g., leaving the patio to find more evidence $\Rightarrow$ another round of updating.

Bayesian updating is iterative and self-consistent.