Normal Dynamic Linear Models (NDLM) & Extensions

Class Notes

Contents

1 Normal Dynamic Linear Model (NDLM)

1.1 General setup

Let the observed data be $\{y_1, \ldots, y_T\}$ and the design matrices be $\{F_1, \ldots, F_T\}$. The NDLM is defined by the system of equations

$$\begin{cases} y_t = F_t^\top \beta_t + \nu_t, & \nu_t \overset{\text{iid}}{\sim} N(0, V), \\ \beta_t = G_t \beta_{t-1} + w_t, & w_t \sim N(0, W), \end{cases} \tag{1}$$

where the dimensions are:

- y_t: 1×1 (scalar observation),
- F_t: $p \times 1$ (design/covariate vector),
- β_t: $p \times 1$ (state/parameter vector),
- G_t: $p \times p$ (evolution matrix),
- w_t: $p \times 1$,
- W: $p \times p$ (evolution variance),
- V: 1×1 (observation variance, denoted R^+ on the board).

In most applications the evolution matrix is constant: $G_t = I_p$.

1.2 Special cases

1.2.1 Local Level Model (LLM)

$$p = 1, \quad F_t = 1, \quad G_t = 1 \quad \forall t.$$

$$y_t = \beta_t + \nu_t, \qquad \beta_t = \beta_{t-1} + w_t.$$

1.2.2 AR(1) + Noise

$$p = 1, \quad F_t = 1, \quad G_t = \phi \quad \forall t.$$

$$y_t = \beta_t + \nu_t, \qquad \beta_t = \alpha + \phi\, \beta_{t-1} + w_t.$$

1.2.3 Linear Growth Model

The observation and state equations are

$$y_t = \beta_t + \gamma_t + \nu_t,$$

$$\beta_t = \beta_{t-1} + \gamma_{t-1} + w_t^{(\beta)}, \qquad \gamma_t = \gamma_{t-1} + w_t^{(\gamma)}.$$

In matrix form, with state vector $\theta_t = (\beta_t, \gamma_t)^\top$:

$$F_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad G_t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \forall t,$$

$$y_t = (1 \;\; 0) \begin{pmatrix} \beta_t \\ \gamma_t \end{pmatrix} + \nu_t,$$

$$\begin{pmatrix} \beta_t \\ \gamma_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_{t-1} \\ \gamma_{t-1} \end{pmatrix} + \begin{pmatrix} w_t^{(\beta)} \\ w_t^{(\gamma)} \end{pmatrix}.$$

2 Extensions to Non-Normal Observations: DGLMs

2.1 From Normality to the Exponential Family

The Gaussian assumption in (1) can be relaxed. Rather than $\nu_t \sim N(0, V)$, we allow y_t to belong to any member of the **exponential family**: Gamma, Beta, Binomial, Poisson, Normal, etc.

The observation equation in (1) then becomes an equivalence between a linear predictor and the conditional mean:

$$y_t = F_t^\top \beta_t + \nu_t \iff y_t \mid F_t, \beta_t \sim N\big(F_t^\top \beta_t,\, V\big),$$

so the conditional expectation is

$$\boxed{E(y_t \mid F_t, \beta_t) = F_t^\top \beta_t.} \tag{2}$$

The evolution equation remains

$$\beta_t = G_t \, \beta_{t-1} + w_t, \qquad w_t \sim N(0, W).$$

2.2 Example 1 — Poisson Dynamic Model

Setup: $y_1, \ldots, y_T \sim \text{Poisson}(\lambda_t)$, with $\lambda_t > 0$.

$$\text{Ex 1:} \quad y_t \mid \lambda_t \sim \text{Poi}(\lambda_t).$$

Link function (log link):

$$\log(\lambda_t) = F_t^\top \beta_t \iff \boxed{\lambda_t = e^{F_t^\top \beta_t}.}$$

Evolution:

$$\beta_t = G_t \, \beta_{t-1} + w_t, \qquad w_t \sim N(0, W).$$

2.3 Example 2 — Dynamic Logistic Regression (Bernoulli)

Setup: $y_1, \ldots, y_T \sim \text{Bernoulli}(\pi_t)$, with $\pi_t \in (0, 1)$.

$$\text{Ex 2:} \quad y_t \mid \pi_t \sim \text{Ber}(\pi_t), \qquad E(y_t \mid \pi_t) = \pi_t.$$

Linear predictor:

$$\pi_t = F_t^\top \beta_t \quad \text{(written loosely; the proper form uses the link below).}$$

Logit link (Dynamic Logistic Regression):

$$\log\left(\frac{\pi_t}{1-\pi_t}\right) = F_t^\top \beta_t \iff \pi_t = \frac{1}{1+e^{-F_t^\top \beta_t}}.$$

Evolution:

$$\beta_t = G_t\, \beta_{t-1} + w_t, \qquad w_t \sim N(0, W).$$

2.3.1 Derivation of the likelihood

The Bernoulli probability mass function is

$$p(y_t \mid \pi_t) = \pi_t^{y_t}(1-\pi_t)^{1-y_t}.$$

Substituting $\pi_t = \left(1 + e^{-F_t^\top \beta_t}\right)^{-1}$:

$$\begin{aligned}
p(y_t \mid \pi_t) &= \left(1 + e^{-F_t^\top \beta_t}\right)^{-y_t} \left(1 - \frac{1}{1+e^{-F_t^\top \beta_t}}\right)^{1-y_t} \\
&= \left(1 + e^{-F_t^\top \beta_t}\right)^{-y_t} \left(\frac{e^{-F_t^\top \beta_t}}{1+e^{-F_t^\top \beta_t}}\right)^{1-y_t} \\
&= \frac{\left(1 + e^{-F_t^\top \beta_t}\right)^{y_t} e^{-F_t^\top \beta_t}}{\left(1 + e^{-F_t^\top \beta_t}\right)^{y_t} \left(1 + e^{-F_t^\top \beta_t}\right)} \cdot \left(\frac{e^{-F_t^\top \beta_t}}{1+e^{-F_t^\top \beta_t}}\right)^{-y_t}.
\end{aligned} \tag{3}$$

3 Exponential Family Form for the Poisson

3.1 Poisson pmf

For $Y \mid \lambda \sim \text{Poi}(\lambda)$, $y = 0, 1, 2, \ldots$:

$$P(y \mid \lambda) = \frac{\lambda^y\, e^{-\lambda}}{y!}.$$

3.2 Exponential family representation

Writing this in the canonical exponential-family form $\frac{1}{a(y)} \exp\bigl(y\,\theta - b(\theta)\bigr)$:

$$P(y \mid \lambda) = \frac{1}{y!} \exp\bigl(y \log\lambda - \lambda\bigr) = \frac{1}{a(y)} \exp\bigl(y\,\theta + c(\theta)\bigr),$$

where $\theta = \log\lambda$ is the *natural parameter* and $a(y) = y!$ is the base measure.

4 Historical Timeline of Dynamic Models

Period	Acronym	Description
∼1850–1970	NLM	Nonlinear Models
1960/1970	NDLM / SSM	Normal Dynamic Linear Model / State Space Models
1970/1971	–	Metropolis–Hastings (Monte Carlo method, 1954–1990)
1970/1980	GLM	Generalized Linear Models (exponential family unification)
1985/1995	DGLM	Dynamic Generalized Linear Models (West, Harrison, Migón, 1985)
1988/1995	MCMC	Gibbs Sampling, Data Augmentation, Metropolis–Hastings
1995–2000	MCMC for DGLM	MCMC applied to Dynamic GLMs

5 Nonlinear Dynamic Model (Nonlinear DM)

A **Nonlinear DM** example with state $x_t \in \mathbb{R}$:

5.1 Observation equation

$$y_t = \frac{x_t^2}{20} + v_t, \qquad v_t \sim N(0,\, \sigma^2).$$

5.2 State equation

$$x_t = \alpha\, x_{t-1} + \beta\left(\frac{x_{t-1}}{1 + x_{t-1}^2}\right) + \gamma \cos(0.2\, t) + w_t, \qquad w_t \sim N(0,\, \tau^2).$$

5.3 Parameter space and priors

The parameter vector is

$$\theta = \big(\sigma^2,\, \tau^2,\, \alpha,\, \beta,\, \gamma\big) \;\in\; (\mathbb{R}^+)^2 \times \mathbb{R}^3.$$

The latent state path is $\mathbf{x} = (x_1, \ldots, x_T)$.

Prior factorisation:

$$p(\theta) = p(\sigma^2)\, p(\alpha, \beta, \gamma \mid \sigma^2)\, p(\tau^2).$$

Initial state prior:

$$x_0 \sim N(m_0,\, C_0).$$