Dynamic GLMs

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Outline

Dynamic generalized linear model

DGLMs were introduced by West, Harrison and Migon (1985).

The model is based on the exponential family of models:

$$
\begin{aligned}
f(y_t|\theta_t) &= a(y_t)\exp\{y_t\theta_t + b(\theta_t)\} \\
E(y_t|\theta_t) &= \mu_t \\
g(\mu_t) &= F_t'\beta_t \\
\beta_t &= G_t\beta_{t+1} + w_t
\end{aligned}
$$

with $w_t \sim N(0, W_t)$ and link function g.

The model is completed with a prior $\beta_1 \sim N(a, R)$.

It combines the prior specification of normal dynamic models with the observational structure of generalized linear models.

Dynamic binomial regression

Dynamic logistic regression with a series of binomial observations y_t with respective success probabilities π_t dynamically related to explanatory variables $x = (x_1, \ldots, x_d)'$ through the logistic link $\text{logit}(\pi_t) = x_t'\beta_t$.

$$\begin{aligned} y_t &\sim Binomial(n_t, \pi_t) \\ \text{logit}(\pi_t) &= x_t'\beta_t \\ \beta_t &= G_t\beta_{t+1} + w_t, \end{aligned}$$

with $w_t \sim N(0, W_t)$.

Dynamic Poisson regression

Poisson counts with means λ_t dynamically related through multiplicative perturbations $\lambda_t = \lambda_{t-1} w_t^*$. After a logarithmic transformation, one obtains $\log\,\lambda_t = \log\,\lambda_{t-1} + w_t$ with $w_t = \log\,w_t^*$.

$$
\begin{aligned}
y_t &\sim Poi(\lambda_t) \\
\log(\lambda_t) &= x_t'\beta_t \\
\beta_t &= G_t\beta_{t+1} + w_t,
\end{aligned}
$$

with $w_t \sim N(0, W_t)$.

Posterior inference via MCMC

Assuming that the variances of the system disturbances are constant, the model parameters are given by the state parameters $\beta = (\beta_1, \ldots, \beta_n)'$ and the system variance $W = \Phi^{-1}$.

The model is specified with the observation and system equations and completed with the independent prior distributions $\beta_1 \sim N(a, R)$ and $\Phi \sim W(n_W/2, n_W S_W/2)$.

The posterior distribution is given by

$$\pi(\beta, \Phi) \propto \prod_{t=1}^{n} f(y_t|\beta_t) \prod_{i=2}^{n} p(\beta_t|\beta_{t-1}, \Phi)\, p(\beta_1)p(\Phi)\ .$$

Full conditional for Φ

$$
\begin{aligned}
\pi_\Phi(\Phi) &\propto \prod_{t=2}^{n} p(\beta_t|\beta_{t-1},\Phi)\, p(\Phi) \\
&\propto \prod_{t=2}^{n} |\Phi|^{1/2} \exp\left\{-\frac{1}{2}\mathrm{tr}[(\beta_t - G_t\beta_{t-1})(\beta_t - G_t\beta_{t-1})'\Phi]\right\} \\
&\times |\Phi|^{[n_W-(p+1)]/2} \exp\left\{-\frac{1}{2}\mathrm{tr}(n_W S_W \Phi)\right\} \\
&\propto |\Phi|^{[n_W^*-(d+1)]/2} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[(n_W^* S_W^*)\,\Phi\right]\right\} .
\end{aligned}
$$

that is the density of the $W(n_W^*/2, n_W^* S_W^*/2)$ distribution with

$$
\begin{aligned}
n_W^* &= n_W + n - 1 \\
n_W^* S_W^* &= n_W S_W + \sum_{t=2}^{n} (\beta_t - G_t\beta_{t-1})(\beta_t - G_t\beta_{t-1})'
\end{aligned}
$$

Full conditionals for β

For block β

$$
\begin{aligned}
\pi_\beta(\beta) &\propto \prod_{t=1}^n f(y_t|\beta_t) \prod_{t=2}^n p(\beta_t|\beta_{t-1},\Phi)\, p(\beta_1) \\
&\propto \exp\left\{\sum_{t=1}^n [y_t\theta_t + b(\theta_t)] - \frac{1}{2}\sum_{t=1}^n (\beta_t - G_t\beta_{t-1})'\Phi(\beta_t - G_t\beta_{t-1})\right\}.
\end{aligned}
$$

For block β_t, $t = 2, \ldots, n-1$

$$
\begin{aligned}
\pi_t(\beta_t) &\propto f(y_t|\beta_t)\, p(\beta_t|\beta_{t-1},\Phi) p(\beta_{t+1}|\beta_t,\Phi) \\
&\propto \exp\{y_t\theta_t + b(\theta_t)\} \exp\left\{-\frac{1}{2}\left[(\beta_t - G_t\beta_{t-1})'\Phi(\beta_t - G_t\beta_{t-1})\right.\right. \\
&+ \left.\left.(\beta_{t+1} - G_{t+1}\beta_t)'\Phi(\beta_{t+1} - G_{t+1}\beta_t)\ \right]\right\}.
\end{aligned}
$$

Similar results follow for blocks β_1 and β_n.

Sampling schemes

Knorr-Held and Besag (1998) and Knorr-Held (2000) suggested the use of independence chains with prior proposals.

Shephard and Pitt (1997) used independence chains with proposals based on both prior and a normal approximation to the likelihood.

Migon *et al.* (2013) used independence normal proposals for the block β with moments given by the approximation of West, Harrison and Migon (1985).

Singh and Roberts (1982) and Fahrmeir and Wagenpfeil (1997) extended to the dynamic setting the method of mode evaluation for static regression.

An alternative previously discussed is the reparametrization in terms of the system disturbances w_t (Gamerman, 1998)

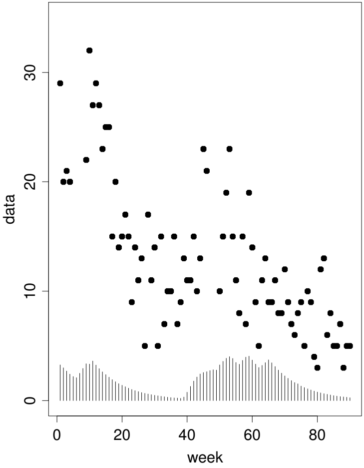
Example: Advertising awareness

Samples of $n_t = 66$ people were selected at random every week for an opinion poll and asked whether they remembered having seen a given advertising campaign on TV. A weekly cumulative measure of campaign expenditure was constructed.

Following Migon and Harrison (1985), the model used for this problem was a dynamic logistic regression

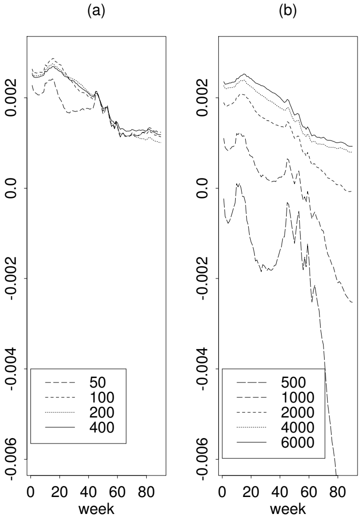
$$
\begin{aligned}
y_t &\sim bin(n_t, \pi_t) \\
\mu_t &= n_t \pi_t \\
\text{logit}(\pi_t) &= \beta_{1t} + \beta_{2t} x_t \\
\beta_t | \beta_{t-1} &\sim N(\beta_{t-1}, W)
\end{aligned}
$$

Data

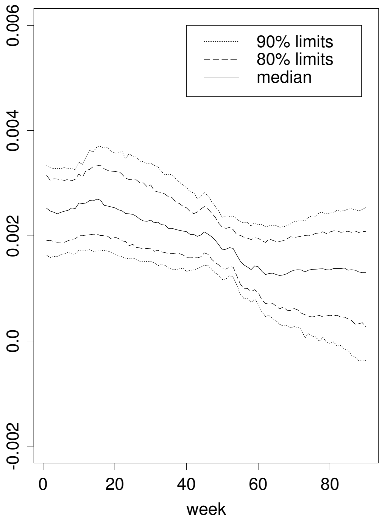


MCMC schemes

Average trajectory of β_{2t} in 500 parallel chains with number of iterations for sampling from: (a) system disturbances; (b) state parameters.



Expenditure coefficient β_{2t}



Example: Nonlinear dynamic model

Let y_t, for $t = 1, \ldots, n$, be generated by the following nonlinear dynamic model

$$
\begin{aligned}
(y_t | x_t, \sigma^2) &\sim N(x_t^2/20, \sigma^2) \\
(x_t | x_{t-1}, \theta, \tau^2) &\sim N(G'_{x_{t-1}}\theta, \tau^2) \\
x_0 &\sim N(m_0, C_0)
\end{aligned}
$$

where $G'_{x_t} = \left(x_t, x_t/(1 + x_t^2), \cos(1.2t)\right)$ and $\theta = (\alpha, \beta, \gamma)'$.

Prior distribution

$$
\begin{aligned}
\sigma^2 &\sim IG(n_0/2, n_0\sigma_0^2/2) \\
\theta|\tau^2 &\sim N(\theta_0, \tau^2 V_0) \\
\tau^2 &\sim IG(\nu_0/2, \nu_0\tau_0^2/2)
\end{aligned}
$$

Sampling $(\theta, \sigma^2, \tau^2 | x_{0:n}, y^n)$

Let $y^n = (y_1, \ldots, y_n)$ and $x_{0:n} = (x_0, \ldots, x_n)'$.

It follows that

$$
\begin{aligned}
(\theta, \tau^2 | x_{0:n}) &\sim N(\theta_1, \tau^2 V_1) IG(\nu_1/2, \nu_1 \tau_1^2/2) \\
(\sigma^2 | y^n, x^n) &\sim IG(n_1/2, n_1 \sigma_1^2/2)
\end{aligned}
$$

where $\nu_1 = \nu_0 + n$, $n_1 = n_0 + n$,

$$
\begin{aligned}
Z &= (G_{x_0}, \ldots, G_{x_{n-1}})' \\
V_1^{-1} &= V_0^{-1} + Z'Z \\
V_1^{-1}\theta_1 &= V_0^{-1}\theta_0 + Z' x_{1:n} \\
\nu_1 \tau_1^2 &= \nu_0 \tau_0^2 + (y - Z\theta_1)'(y - Z\theta_1) + (\theta_1 - \theta_0)' V_0^{-1} (\theta_1 - \theta_0) \\
n_1 \sigma_1^2 &= n_0 \sigma_0^2 + \sum_{t=1}^{n} (y_t - x_t^2/20)^2
\end{aligned}
$$

Sampling $(x_t|x_{-t},\theta,\sigma^2,\tau^2,y^n)$

Let $x_{-t}=(x_0,\ldots,x_{t-1},x_{t+1},\ldots,x_n)$, for $t=1,\ldots,n-1$, $x_{-0}=x^n$, $x_{-n}=x_{0:(n-1)}$, $y_0=\emptyset$, and $\psi=(\theta,\sigma^2,\tau^2)$.

For $t=0$

$$p(x_0|x_{-0},y_0,\psi)\propto f_N(x_0;m_0,C_0)f_N(x_1;G'_{x_0}\theta,\tau^2)$$

For $t=1,\ldots,n-1$

$$\begin{aligned}p(x_t|x_{-t},y_t,\psi) &\propto f_N(y_t;x_t^2/20,\sigma^2)\\ &\times f_N(x_t;G'_{x_{t-1}}\theta,\tau^2)\\ &\times f_N(x_{t+1};G'_{x_t}\theta,\tau^2)\end{aligned}$$

For $t=n$

$$p(x_n|x_{-n},y_n,\psi)\propto f_N(y_n;x_n^2/20,\sigma^2)f_N(x_n;G'_{x_{n-1}}\theta,\tau^2)$$

Metropolis-Hastings algorithm

A simple random walk Metropolis algorithm with tuning variance v_x^2 would work as follows. For $t = 0, \ldots, n$

1. Current state: $x_t^{(j)}$
2. Sample x_t^* from $N(x_t^{(j)}, v_x^2)$
3. Compute the acceptance probability

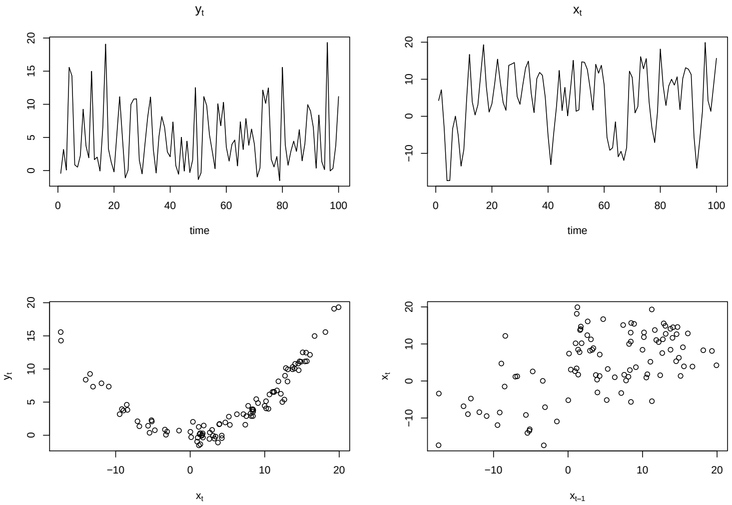
$$\alpha = \min\left\{1, \frac{p(x_t^*|x_{-t}, y_t, \psi)}{p(x_t^{(j)}|x_{-t}, y_t, \psi)}\right\}$$

4. New state:

$$x_t^{(j+1)} = \begin{cases} x_t^* & \text{w. p. } \alpha \\ x_t^{(j)} & \text{w. p. } 1-\alpha \end{cases}$$

Simulated exercise

We simulated $n = 100$ observations based on $\theta = (0.5, 25, 8)'$, $\sigma^2 = 1$, $\tau^2 = 10$ and $x_0 = 0.1$.



Prior hyperparameters

- $x_0 \sim N(m_0, C_0)$

$$m_0 = 0.0 \quad \text{and} \quad C_0 = 10$$

- $\theta|\tau^2 \sim N(\theta_0, \tau^2 V_0)$

$$\theta_0 = (0.5, 25, 8)' \quad \text{and} \quad V_0 = \text{diag}(0.0025, 0.1, 0.04)$$

- $\tau^2 \sim IG(\nu_0/2, \nu_0\tau_0^2/2)$

$$\nu_0 = 6 \quad \text{and} \quad \tau_0^2 = 20/3$$

  such that $E(\tau^2) = \sqrt{V(\tau^2)} = 10.$

- $\sigma^2 \sim IG(n_0/2, n_0\sigma_0^2)$

$$n_0 = 6 \quad \text{and} \quad \sigma_0^2 = 2/3$$

  such that $E(\sigma^2) = \sqrt{V(\sigma^2)} = 1.$

MCMC setup

- Metropolis-Hastings tuning parameter

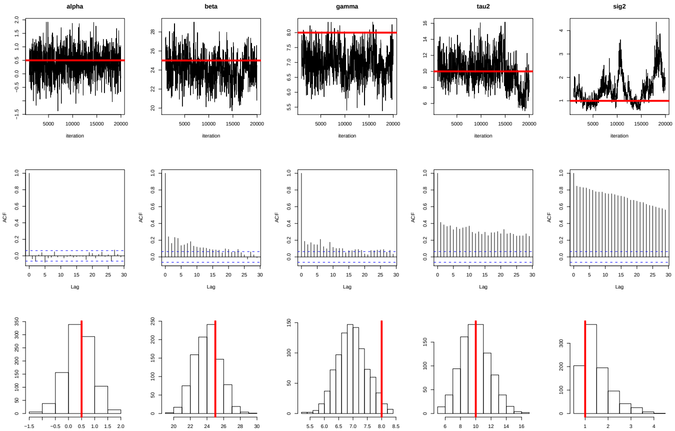
$$v_x^2 = (0.1)^2$$

- Burn-in period, step and MCMC sample size

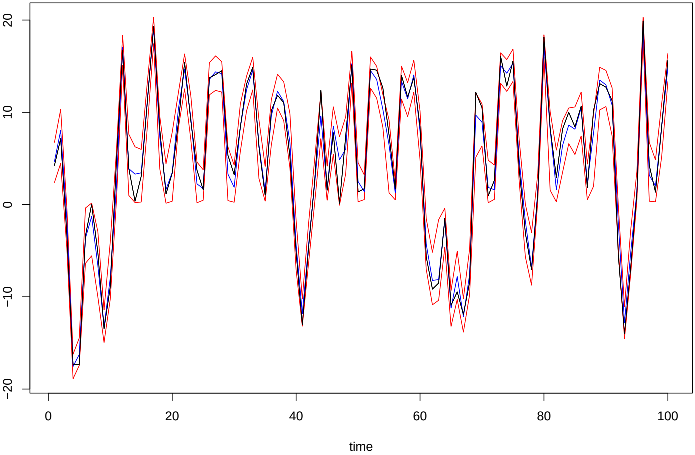
$$M_0 = 1,000 \quad L = 20 \quad M = 950 \;\Rightarrow\; 20,000 \text{ draws}$$

- Initial values
  - $\theta = (0.5, 25, 8)'$
  - $\tau^2 = 10$
  - $\sigma^2 = 1$
  - $x_{0:n} = x_{0:n}^{\text{true}}$

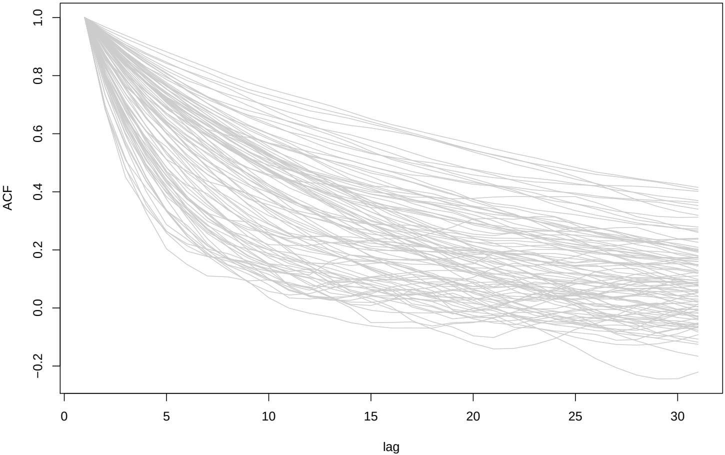
Parameters



States



States



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