Monty Hall

A Simple Bayesian solution

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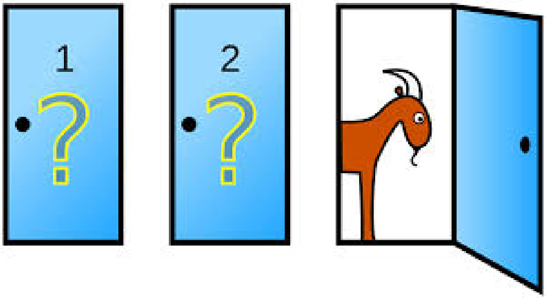
The Monty Hall Problem[1]

Game setup: The host of a game show, Monty Hall, invites you to the stage and shows you three doors. There is a brand new Ferrari behind one of the three doors. Goats are behind the other two doors.



[1]Illustration taken from Wikipedia.

Let's Make a Deal

First step
Monty asks you to select a door and you choose, say, door A (doors are labeled A, B and C).

Second step: Monty opens one of the other doors, say B, with a goat.

Third step: Monty offers you one of two possible decisions:

- $\mathcal{D}_A$: Maintain door A as your selection.
- $\mathcal{D}_C$: Choose door C as your selection.

Which decision would you make? $\mathcal{D}_A$ or $\mathcal{D}_C$?

Bayesian approach

Prior: The Ferrari is behind one of the three doors with probability $1/3$. That is the nature of the game, randomly selecting a door and placing a Ferrari behind it and goats behind the other two doors. Let Y be the random variable that states which door you selected, $Y \in \{A, B, C\}$. Therefore

$$Pr(Y = A) = Pr(Y = B) = Pr(Y = C) = \frac{1}{3}$$

Likelihood: Let X be the random variable that indicates which door Monty opens. Since we know that $X = B$, we can compute the likelihoods

$$\begin{array}{rcll} Pr(X = B|Y = A) & = & \frac{1}{2} & \text{(When } Y = A\text{, Monty can pick } B \text{ or } C\text{)} \\ Pr(X = B|Y = B) & = & 0 & \text{(When } Y = B\text{, Monty cannot pick } B\text{)} \\ Pr(X = B|Y = C) & = & 1 & \text{(When } Y = C\text{, Monty has to pick } C\text{)} \end{array}$$

Which decision?

Notice that $P(Y = B|X = B) = 0$, since there is zero chance of finding a Ferrari behind the door Monty opened.

Therefore, it is easy to see that you will make the decision $\mathcal{D}_C$ over the decision $\mathcal{D}_A$ if

$$Pr(Y = C|X = B) > Pr(Y = A|X = B).$$

These quantities (*Posterior*) are now easily obtained via Bayes Theorem,

$$\begin{aligned} Pr(Y = C|X = B) &= \frac{Pr(X = B|Y = C)Pr(Y = C)}{Pr(X = B)} = \frac{1 \times 1/3}{Pr(X = B)} = \frac{1/3}{Pr(X = B)} \\ Pr(Y = A|X = B) &= \frac{Pr(X = B|Y = A)Pr(Y = A)}{Pr(X = B)} = \frac{1/2 \times 1/3}{Pr(X = B)} = \frac{1/6}{Pr(X = B)} \end{aligned}$$

$P(Y = C|X = B) = 2Pr(Y = A|X = B) \Longrightarrow$ Swap to door C!