Take home midterm exam

PhD in Business Economics

Course: Bayesian Econometrics

Professor: Hedibert Freitas Lopes

Start: 10am, October 5th, 2024

End: 10pm, October 7th, 2024 (60 hours later!)

Simple hierarchical regression on time: Souza (1999)[1] considers a number of hierarchical and dynamic models to describe the nutritional pattern of pregnant women (data appear at the end of this document). The data depicted in Figure 5.5 (page 161) of Gamerman and Lopes (2006) consist of the weight gains (y_{ij}s) of $I = 68$ pregnant women at 5 to 7 visits (x_{ij}s) to the Instituto de Puericultura e Pediatria Martagão Gesteira from the Universidade Federal do Rio de Janeiro. One of the simplest models she adopted was the simple hierarchical regression on time where

$$\begin{aligned} y_{ij}|\alpha_i, \beta_i, \sigma^2 &\sim N(\alpha_i + \beta_i x_{ij}, \sigma_y^2) \\ (\alpha_i, \beta_i)'|\alpha, \beta &\sim N((\alpha, \beta)', \text{diag}(\sigma_\alpha^2, \sigma_\beta^2)) \\ (\alpha, \beta) &\sim N(0_2, 1000 I_2), \end{aligned}$$

for $i = 1, \ldots, I = 68$, $n = n_1 + \cdots n_I = 427$. The variance parameters $\sigma_y^2, \sigma_\alpha^2$ and σ_β^2 are all *a priori* independent $IG(0.001, 0.001)$. For (a)-(d) below, ignore the missing structure for those women that do not have either or both visits 6 and 7 recorded. For instance, the first four women have missed the last visit, so their contribution are (x_{ij}, y_{ij}) for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5, 6$.

(a) Fit, by OLS, the 68 individual models for comparison with (c) and (d).

- I already run the 68 regressions and summarize the $\hat{\alpha}_i$s and $\hat{\beta}_i$s by quartiles.
- α_is: $(-6.744, -4.260, -2.33)$ and β_is: $(0.395, 0.450, 0.525)$.
- The quartiles of the 68 OLS residual standard errors are $(0.735, 0.955, 1.157)$.
- These results might guide your own computations.

(b) Fit, by OLS, the pooled model, where $(\alpha_i, \beta_i) = (\alpha, \beta)$ for all I, also for comparison.

- The OLS estimates of α and β are -4.526 and 0.476, respectively, while the overall residual standard error is 3.612. Notice that the overall residual standard error is three times as big as its counterpart from the 68 individual regressions, despite the fact that those regressions are based only on 5, 6 or 7 data points.

[1] Aparecida D. P. Souza (1999) Approximate Methods in Bayesian Dynamic Hierarchical Models, unpublished Ph.D. Thesis, COPPE-UFRJ (in Portuguese).

(c) Show that all full conditionals of the above simple hierarchical regression on time are of known form. More precisely, they are all either Gaussian or Inverse-Gamma densities, therefore suitable for a standard Gibbs Sampler scheme, i.e. the full conditional distributions of σ_y^2, σ_α^2 and σ_β^2 are all Inverse-Gammas, while the full conditional distributions of (α, β) and (α_i, β_i), for $i = 1, \ldots, 68$, are bivariate Gaussians.

(d) Implement the Gibbs sampler, and obtain posterior inference for the above simple hierarchical regression on time. Compare the results with those found in (a) and (b).

(e) Obtain posterior medians and 95% credibility intervals for the y_{i7}s that are not observed for three of the first four women, i.e. $i = 1, 2, 4$. Plot the histograms of $p(y_{i7}|x_{i7}, \text{data})$, for $i = 1, 2, 4$.

Throughout, please do comment thoroughly your findings and impressions.

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