Take-home

Full conditional distribution of (α,β)

$\theta=(\alpha,\beta)^T \quad p(\theta)\propto \exp\{-\frac{1}{2}\theta^T A_0^{-1}\theta\} \qquad A_0=1000 I_2$

$\theta_i=(\alpha_i,\beta_i)^T \quad p(\theta_1,\dots,\theta_I|\theta,\Sigma)\propto \exp\{-\frac{1}{2}\sum_{i=1}^{I}(\theta_i-\theta)^T\Sigma^{-1}(\theta_i-\theta)\}$

$\Sigma=\begin{pmatrix}\sigma_\alpha^2 & 0\\ 0 & \sigma_\beta^2\end{pmatrix}$

$$\Rightarrow p(\theta|\dots)\propto \exp\{-\frac{1}{2}\left(\theta^T A_0^{-1}\theta + I.\theta^T\Sigma^{-1}\theta - 2\theta\Sigma^{-1}\sum_{i=1}^{I}\theta_i\right)\}$$

$$\propto \exp\{-\frac{1}{2}\left[\theta^T(A_0^{-1}+I\Sigma^{-1})\theta - 2\theta^T\Sigma^{-1}\sum_{i=1}^{I}\theta_i\right]\}$$

$$\Rightarrow [\alpha,\beta] \sim N\left[(A_0^{-1}+I\Sigma^{-1})^{-1}\Sigma^{-1}\sum_{i=1}^{I}\theta_i \; ; \; (A_0^{-1}+I\Sigma^{-1})^{-1}\right]$$

Note: $A_0^{-1}+I\Sigma^{-1}=\begin{pmatrix}\frac{1}{1000} & 0\\ 0 & \frac{1}{1000}\end{pmatrix}+\begin{pmatrix}I/\sigma_\alpha^2 & 0\\ 0 & I/\sigma_\beta^2\end{pmatrix}=\begin{bmatrix}\frac{1}{1000}+I/\sigma_\alpha^2 & 0\\ 0 & \frac{1}{1000}+I/\sigma_\beta^2\end{bmatrix}$

Full conditional distribution of σ_y^2

$\sigma_y^2\sim I(a_0,b_0)$

$$p(\sigma_y^2|\dots)\propto(\sigma_y^2)^{-(a_0+1)}e^{-\frac{b_0}{\sigma_y^2}}\times(\sigma_y^2)^{-\frac{M}{2}}e^{-\frac{1}{2\sigma_y^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-\alpha_i-\beta_i x_{ij})^2}$$

$$\Rightarrow [\sigma_y^2]\sim IG\left(a_0+\frac{n}{2};\; b_0+\frac{\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-\alpha_i-\beta_i x_{ij})^2}{2}\right)$$

Full conditional distributions of σ_α^2 and σ_β^2 —

$$\sigma_\alpha^2 \sim IG(c_0, d_0) \quad \& \quad \sigma_\beta^2 \sim IG(e_0, f_0)$$

$$p(\sigma_\alpha^2 | \dots) \propto (\sigma_\alpha^2)^{-(c_0+1)} e^{-d_0/\sigma_\alpha^2} \times (\sigma_\alpha^2)^{-\frac{I}{2}} e^{-\frac{1}{2\sigma_\alpha^2}\sum_{i=1}^{I}(\alpha_i - \alpha)^2}$$

$$\propto (\sigma_\alpha^2)^{-(c_0+I/2)} \exp\left\{-\frac{d_0 + \sum_{i=1}^{I}(\alpha_i-\alpha)^2}{\sigma_\alpha^2}\right\}$$

$$\Rightarrow [\sigma_\alpha^2] \sim IG\left(c_0 + I/2 \, ; \, d_0 + \frac{\sum_{i=1}^{I}(\alpha_i - \alpha)^2}{2}\right)$$

$$\Rightarrow [\sigma_\beta^2] \sim IG\left(e_0 + I/2 \, ; \, f_0 + \frac{\sum_{i=1}^{I}(\beta_i - \beta)^2}{2}\right)$$

Full conditional distributions of $\theta_i = (\alpha_i, \beta_i)^T$

$$p(\theta_i | \dots) \propto \exp\left\{-\frac{1}{2}(\theta_i - \theta)^T \Sigma^{-1} (\theta_i - \theta)\right\} \exp\Big\{-\frac{1}{2\sigma_y^2} \underbrace{\sum_{j=1}^{m_i}(y_{ij} - \alpha_i - \beta_i x_{ij})^2}_{(y_i - X_i\theta_i)^T(y_i - X_i\theta_i)}\Big\}$$

$$y_i = \begin{pmatrix} y_{i1} \\ \vdots \\ y_{im_i} \end{pmatrix} \quad X_i = \begin{pmatrix} 1 & x_{i1} \\ 1 & x_{i2} \\ \vdots & \\ 1 & x_{im_i} \end{pmatrix}$$

$$\Rightarrow p(\theta_i | \dots) \propto \exp\left\{-\frac{1}{2\sigma_y^2}\left[\theta_i^T \Sigma^{-1}\theta_i - 2\theta_i^T\Sigma^{-1}\theta + \theta_i^T X_i^T X_i \theta_i - 2\theta_i^T X_i^T y_i\right]\right\}$$

$$\Rightarrow (\theta_i | \dots) \sim N\left[\left(\Sigma^{-1} + X^TX/\sigma_y^2\right)^{-1}\left(\Sigma^{-1}\theta + X_i^T y_i/\sigma_y^2\right); \left(\Sigma^{-1} + X^TX/\sigma_y^2\right)\right]$$