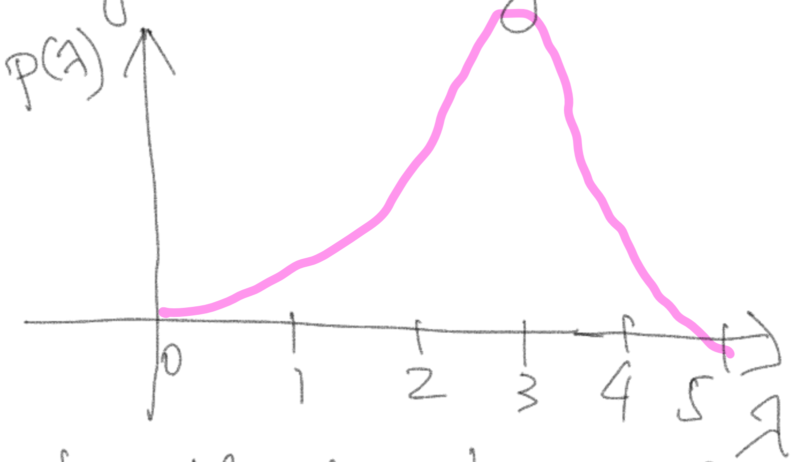
Poisson model with some pre-existing knowledge about λ.

Suppose



An easy to handle density is the Exponential density

$$P(\lambda|a) = e^{-a\lambda} \qquad \lambda \geq 0$$

with $\quad E(\lambda) = 1/a$.

$$V(\lambda) = 1/a^2$$

If we believe $E(\lambda) \approx 2$ and are happy with $V(\lambda) \approx 4$, then $a = 1/2$ is a reasonable choice.

The Bayesian approach allows us to combine different sources of information:

Likelihood : $L(\lambda | y_1 \dots y_n) \propto \lambda^{S_n} e^{-n\lambda}$

Prior : $p(\lambda) \propto e^{-a\lambda}$

Then

$$\text{Posterior} \propto \text{Likelihood} \times \text{prior}$$

$$\propto \lambda^{S_n} e^{-(a+n)\lambda}$$

This posterior has the kernel of a Gamma distribution with parameters $(S_n + 1)$ and $(a+n)$, denoted here by

$$(\lambda | y_1, \dots, y_n) \sim G(S_n + 1 ; a+n).$$

$$\left\lceil X \sim G(\alpha,\beta) \Leftrightarrow p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \right\rceil$$

$$E(X) = \frac{\alpha}{\beta} ; \quad V(X) = \frac{\alpha}{\beta^2} .$$

$$\left\lfloor \mathrm{Mode}(X) = \frac{\alpha-1}{\beta} ; \ \alpha \geq 1 \right\rfloor$$

Now $$E(\lambda | y_1 \ldots y_n) = \frac{S_n + 1}{a + n} \overset{a=\frac{1}{2}}{=} \frac{S_n+1}{n+\frac{1}{2}}$$

$$\mathrm{Mode}(\lambda | y_1, \ldots, y_n) = \frac{S_n}{a+n} = \frac{S_n}{n+\frac{1}{2}}$$

$$V(\lambda | y_1, \ldots, y_n) = \frac{S_n+1}{(a+n)^2} = \frac{S_n+1}{(n+\frac{1}{2})^2}$$

In our coal mining example $n=41$; $S_n=127$

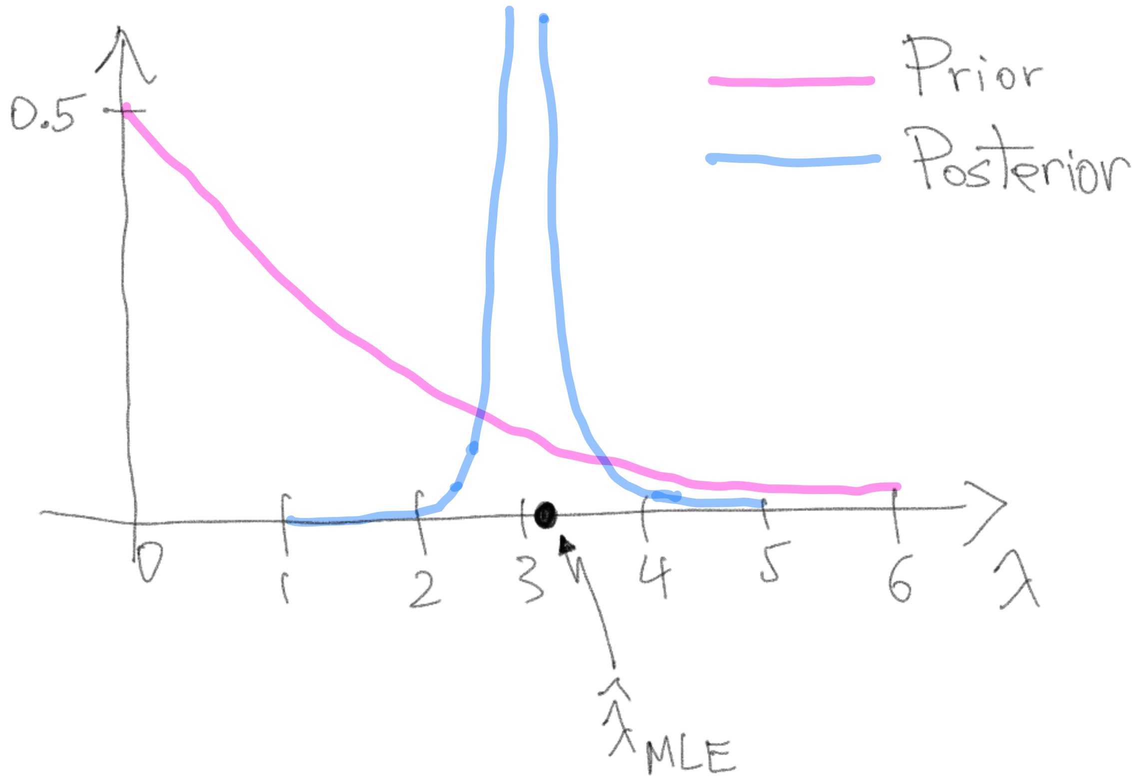
$$E(\lambda | n, S_n) = \frac{128}{41.5} \cong 3.084$$

$$\mathrm{Mode}(\lambda | n, S_n) = \frac{127}{41.5} \cong 3.060$$

$$\sqrt{\mathrm{Var}(\lambda | n, S)} = \frac{\sqrt{128}}{41.5} \cong 0.273$$

Recall that

$$\hat{\lambda}_{MCE} = \frac{S_n}{n} = \frac{127}{41} = 3.098$$



In this example, the prior information is small when compared to the information in the likelihood.

Still $E(\lambda | y_1, \ldots, y_n)$ and $\text{Mode}(\lambda | y_1, \ldots, y_n)$ are a little bit to the left of $\hat{\lambda}_{MLE}$.

This is what we call <u>shrinkage</u>.