Comparing estimators: MSE

Suppose $X_1, \dots, X_n \mid \theta$ iid $\mathcal{U}[0,\theta]$, i.e. uniform between 0 and θ, for $\theta > 0$. We want to compare 4 estimators:

$$\begin{cases} \hat{\theta}_1 = \bar{X} \\ \hat{\theta}_2 = X_{(n)} \end{cases} \qquad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Order statistics: $X_{(1)}, \dots, X_{(n)}$

$$\begin{cases} \hat{\theta}_3 = 2\hat{\theta}_1 \\ \hat{\theta}_4 = \left(\frac{n+1}{n}\right)\hat{\theta}_2 \end{cases}$$

	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$
$E(\hat{\theta})$	$\theta/2$	$\frac{n}{n+1}\theta$	θ	θ
$V(\hat{\theta})$	$\theta^2/12n$	$\frac{n\theta^2}{(n+1)^2(n+2)}$	$\theta^2/3n$	$\frac{\theta^2}{n(n+2)}$

$\hat{\theta}_3$ and $\hat{\theta}_4$ are the unbiased versions of $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively.

To compare the estimators we will use mean squared errors:

$$MSE(\hat{\theta}) = E_{X|\theta}\left\{\left(\hat{\theta} - E(\theta)\right)^2\right\}$$

$$= \underbrace{V(\hat{\theta})}_{\text{variance}} + [\underbrace{E(\hat{\theta}) - \theta}_{\text{Bias}}]^2$$

Obviously, $MSE(\hat{\theta}_i) = V(\hat{\theta}_i)$ for $i = 3, 4$

We can show that

$$MSE(\hat{\theta}_1) = \left(\frac{1+3n}{12n}\right)\theta^2$$

$$MSE(\hat{\theta}_2) = \left(\frac{2}{(n+1)(n+2)}\right)\theta^2$$

Therefore, for $n \geq 2$

$$MSE(\hat{\theta}_4) < MSE(\hat{\theta}_2) < MSE(\hat{\theta}_3) < MSE(\hat{\theta}_1)$$

Let us explore a bit further $\hat{\theta}_1$ and $\hat{\theta}_2$

$\boxed{\hat{\theta}_1 = \bar{X} \text{ is a "natural choice".}}$

Not quite! All X_i's are less than or equal to θ, so most likely $\bar{X}_n$ will be even further away!

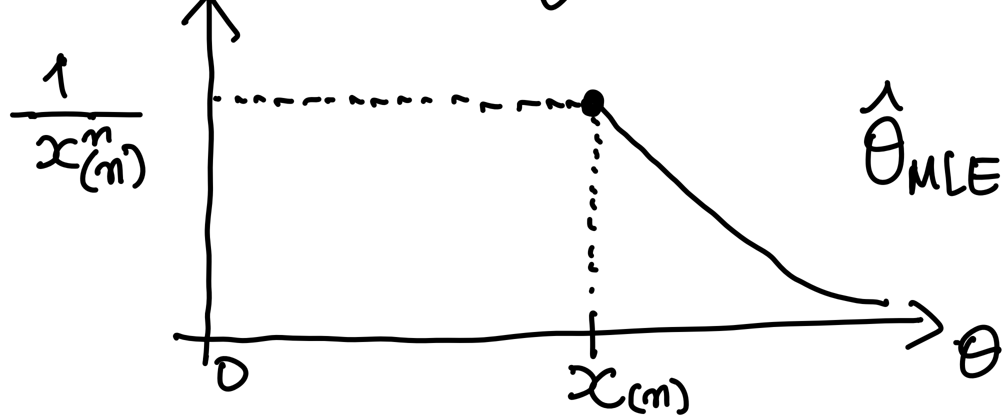
$\boxed{\text{MLE}}$

$$p(x_1, \dots, x_n \mid \theta) = \left(\frac{1}{\theta}\right)^n \mathbb{1}(x_1 \leq \theta) \times \dots \times \mathbb{1}(x_n \leq \theta)$$

$$= \left(\frac{1}{\theta}\right)^n \mathbb{1}(x_{(n)} \leq \theta)$$

where $x_{(n)} = \max(x_1, \dots, x_n)$. So,

$$L(\theta \mid x_1, \dots, x_n) = \frac{1}{\theta^n} \mathbb{1}(\theta \geq x_{(n)})$$



$$\hat{\theta}_{MLE} = x_{(n)}$$