$\boxed{\text{Solução}}$

$$y_1, \dots, y_n \mid \theta \overset{iid}{\sim} Poi(\theta) \qquad \theta > 0$$

$$\theta \sim G(\alpha, \beta) \qquad \alpha, \beta > 0 \qquad S_n = \sum_{i=1}^{n} y_i$$

a) $$p(y_1 \dots y_n) = \int_0^\infty \left[\prod_{i=1}^{n} p(y_i \mid \theta)\right] p(\theta)\, d\theta = \int_0^\infty \frac{\theta^{S_n} e^{-n\theta}}{\prod_{i=1}^{n} y_i!} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}\, d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha) \prod_{i=1}^{n} y_i!} \underbrace{\int \theta^{(\alpha+S_n)-1} e^{-(\beta+n)\theta}\, d\theta}_{\frac{\Gamma(\alpha+S_n)}{(\beta+n)^{\alpha+S_n}}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha) \prod_{i=1}^{n} y_i!} \cdot \frac{\Gamma(\alpha+S_n)}{(\beta+n)^{\alpha+S_n}} \qquad \#$$

b) $$p(\theta \mid y_1, \dots, y_n) \propto \theta^{\alpha-1} e^{-\beta\theta} \theta^{S_n} e^{-n\theta}$$

$$\propto \theta^{(\alpha+S_n)-1} e^{-(\beta+n)\theta} \sim G(\overbrace{\alpha+S_n}^{\alpha_1}, \overbrace{\beta+n}^{\beta_1})$$

c) $$p(y_{n+1} \mid y_1, \dots, y_n) = \int_0^1 p(y_{n+1} \mid \theta)\, p(\theta \mid y_1 \dots y_n)\, d\theta$$

$$= \int_0^1 \frac{\theta^{y_{n+1}} e^{-\theta}}{y_{n+1}!} \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \theta^{\alpha_1 - 1} \beta^{-\beta_1 \theta}\, d\theta =$$

$$= \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \cdot \frac{1}{(y_{n+1})!} \cdot \underbrace{\int_0^1 \theta^{(\alpha_1 + y_{n+1})} e^{-(\beta_1 + 1)\theta}\, d\theta}_{\left[\frac{(\beta_1+1)^{\alpha_1 + y_{n+1}}}{\Gamma(\alpha_1 + y_{n+1})}\right]^{-1}}$$

$$= \frac{\beta_1^{\alpha_1}}{\beta_1^{\alpha_1 + y_{n+1}}} \frac{\Gamma(\alpha_1 + y_{n+1})}{\Gamma(\alpha_1)} \cdot \frac{1}{y_{n+1}!}$$

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