PhD in Business Economics — Advanced Bayesian Econometrics

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Please submit either your file (handwritten or typed) in PDF or HTML. The file must be a single PDF/HTML document for submission to me at `hedibertfl@insper.edu.br`. Students should follow the deadlines for submissions. This homework assignment should be done individually.

AR(1) plus noise

Suppose we observe some time series data y_t for $t = 1, \ldots, n$, jointly denoted by $y_{1:n}$ when needed, and consider the following normal dynamic linear model:

$$
\begin{aligned}
y_t &= \theta_t + \epsilon_t & \epsilon_t \sim N(0, \sigma^2), \qquad (t = 1, \ldots, n),\\
\theta_t &= \alpha + \beta\theta_{t-1} + \omega_t & \omega_t \sim N(0, \tau^2), \qquad (t = 2, \ldots, n),
\end{aligned}
$$

while $\theta_1 \sim N(a_1, R_1)$, for known hyperparameters a_1 and R_1. ϵ_t and ω_{t+h} are uncorrelated for all h.

Now, conditioning on $\gamma = (\alpha, \beta, \tau^2, \sigma^2)$ and $y_{1:n}$, derive the following full conditionals

a) $p(\theta_1|\theta_{-1}, y_{1:n}, \gamma)$

b) $p(\theta_n|\theta_{-n}, y_{1:n}, \gamma)$

c) $p(\theta_t|\theta_{-t}, y_{1:n}, \gamma)$

where $\theta_{-t} = (\theta_1, \ldots, \theta_{t-1}, \theta_{t+1}, \ldots, \theta_n)$. Because of the Markovian structure of the dynamics of θ_t, it is easy to show that

$$
\begin{aligned}
p(\theta_1|\theta_{-1}, y_{1:n}, \gamma) &= p(\theta_1|\theta_2, y_1, \gamma) \propto p(y_1|\theta_1, \gamma)p(\theta_1)p(\theta_2|\theta_1, \gamma)\\
p(\theta_n|\theta_{-n}, y_{1:n}, \gamma) &= p(\theta_n|\theta_{n-1}, y_n, \gamma) \propto p(y_n|\theta_n, \gamma)p(\theta_n|\theta_{n-1}\gamma)\\
p(\theta_t|\theta_{-t}, y_{1:n}, \gamma) &= p(\theta_t|\theta_{t-1}, \theta_{t+1}, y_t, \gamma) \propto p(y_t|\theta_t, \gamma)p(\theta_t|\theta_{t-1}, \gamma)p(\theta_{t+1}|\theta_t, \gamma).
\end{aligned}
$$

Since all densities are Gaussian and linear on θ_t, then all full conditional are also Gaussian. Your job is to simply derive the means and variances of these n Gaussian distribution.

Simulating some data and running the Gibbs sampler

d) Let $n = 100$, $\theta_0 = 0$, $\gamma = (0, 1, 0.25, 1)$, simulate $y_{1:n}$ following an AR(1) plus noise process.

e) Using the derivations from a)-c), with $a_1 = 0$ and $R_1 = 9$, implement the Gibbs sampler and obtain $M = 1,000$ draws, after discarding $M_0 = 1000$ as burn-in, from $p(\theta_1, \ldots, \theta_n|y_{1:n}, \gamma)$. Let us call these draws $\{\theta_1^{(i)}, \ldots, \theta_n^{(i)}\}$. As for initial values, let $\theta_{1:n}^{(0)} = y_{1:n}$. For each $t \in \{1, \ldots, n\}$, obtain the 95% credible interval for θ_t along with its median, i.e. obtain the 2.5th, 50th and 97.5th percentiles.

Below you find my own code for simulating the data:

```
set.seed(12345)
n      = 100
sig    = 1
tau    = 0.25
alpha  = 0
beta   = 1
theta0 = 0
sig2 = sig^2
tau2 = tau^2
theta    = rep(0,n)
theta[1] = rnorm(1,alpha+beta*theta0,tau)
for (t in 2:n)
  theta[t] = rnorm(1,alpha+beta*theta[t-1],tau)
y = rnorm(n,theta,sig)
plot(y)
lines(theta,col=2)
```

Learning about γ and running the complete Gibbs sampler

f) Now, let us assume that (α, β) follows a zero-mean bivariate normal with covariance δI_2, independent of τ^2 and σ^2. Also, let $\sigma^2 \sim IG(a_\sigma, b_\sigma)$ and $\tau^2 \sim IG(a_\tau, b_\tau)$. The hyperparameters are $\delta = 9$, $a_\sigma = b_\sigma = a_\tau = b_\tau = 1$. Derive the additional full conditional distributions:

f.1) $p(\alpha, \beta|\tau^2, \sigma^2, \theta_{1:n}, y_{1:n}) \equiv p(\alpha, \beta|\tau^2, \theta_{1:n})$ - Bivariate Gaussian,

f.2) $p(\tau^2|\alpha, \beta, \theta_{1:n}, y_{1:n}) \equiv p(\tau^2|\alpha, \beta, \theta_{1:n})$ - Inverse Gamma,

f.3) $p(\sigma^2|\alpha, \beta, \theta_{1:n}, y_{1:n}) \equiv p(\sigma^2|\theta_{1:n}, y_{1:n})$ - Inverse Gamma,

You are now ready for the full-blown Gibbs sampler for the AR(1) plus noise model.

g) Compare the 95% credibility intervals for $p(\theta_t|\alpha, \beta, \tau^2, \sigma^2, y_{1:n})$ from d) with $p(\theta_t|y_{1:n})$ from f).