AR(1) plus noise model

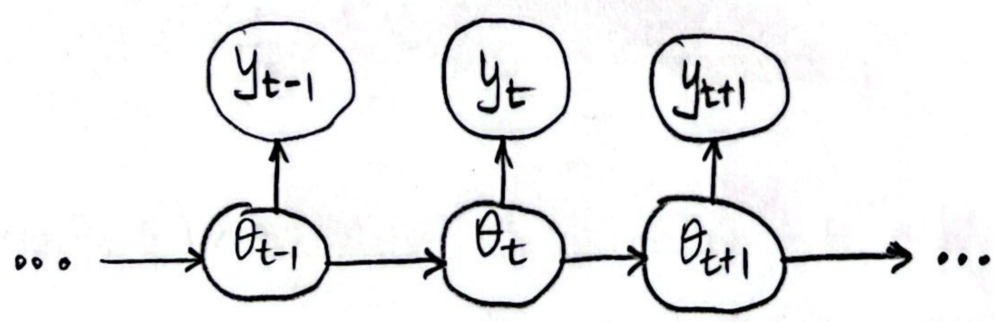
data: $\{y_1, \ldots, y_n\}$

model:
$$y_t = \theta_t + \epsilon_t \qquad \epsilon_t \sim N(0, \sigma^2)$$
$$\theta_t = \alpha + \beta\theta_{t-1} + \omega_t \qquad \omega_t \sim N(0, \tau^2)$$
$$\theta_1 \sim N(a_1, R_1) \qquad \gamma = (\alpha, \beta, \tau^2, \sigma^2)$$

Derive
i) $p(\theta_t \mid \theta_{-t}, y_{1:n}, \gamma) \qquad t = 2, \ldots, n-1$
ii) $p(\theta_1 \mid \theta_{2:n}, y_{1:n}, \gamma)$
iii) $p(\theta_n \mid \theta_{1:(n-1)}, y_{1:n}, \gamma)$

Note: The dynamic/Markovian structure of the evolution of θ_t allows us to write

$t = 2, \ldots, n-1$
$$p(\theta_t \mid \theta_{-t}, y_{1:n}, \gamma) = p(\theta_t \mid \theta_{t-1}, \theta_{t+1}, y_t, \gamma)$$
$$p(\theta_1 \mid \theta_{2:n}, y_{1:n}, \gamma) = p(\theta_1 \mid \theta_2, y_1, \gamma)$$
$$p(\theta_n \mid \theta_{1:(n-1)}, y_{1:n}, \gamma) = p(\theta_n \mid \theta_{n-1}, y_n, \gamma)$$



Simulate data

$\theta_0 = 0 \quad \sigma = 1 \quad \tau = 0.25 \quad \alpha = 0 \quad \beta = 1$

$n = 100 \Rightarrow \{y_1, \ldots, y_n\}$

Run Gibbs sampler for $\theta_{1:n}$

with $a_1 = 0 \quad R_1 = 1$

burn-in 1000
draws 1000
step 1

For $t=2,3,\dots,n-1$

$$p(\theta_t|\theta_{t-1},\theta_{t+1},y_t,\gamma) \propto p(y_t|\theta_t,\gamma)\,p(\theta_t|\theta_{t-1},\gamma)\,p(\theta_{t+1}|\theta_t,\gamma)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{y_t^2}-2y_t\theta_t+\theta_t^2\right)\right\}\exp\left\{-\frac{1}{2\tau^2}\left(\theta_t^2-2\theta_t(\alpha+\beta\theta_{t-1})+\cancel{(\alpha+\beta\theta_{t-1})^2}\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2\tau^2}\left(\cancel{(\theta_{t+1}-\alpha)^2}-2\beta\theta_t(\theta_{t+1}-\alpha)+\beta^2\theta_t^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta_t^2\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}+\frac{\beta^2}{\tau^2}\right)-2\theta_t\left(\frac{y_t}{\sigma^2}+\frac{\alpha+\beta\theta_{t-1}}{\tau^2}+\frac{\beta(\theta_{t+1}-\alpha)}{\tau^2}\right)\right]\right\}$$

$$\Rightarrow (\theta_t|\theta_{t-1},\theta_{t+1},y_t,\gamma) \sim N(m_t,C_t) \qquad C_t^{-1}=\sigma^{-2}+(1+\beta^2)\tau^{-2}$$

$$m_t = C_t\left(\frac{y_t}{\sigma^2}+\frac{\alpha(1-\beta)+\beta(\theta_{t+1}+\theta_{t-1})}{\tau^2}\right)$$

$t=1$

$$p(\theta_1|\theta_2,y_1,\gamma)\propto p(y_1|\theta_1,\gamma)\,p(\theta_2|\theta_1,\gamma)\,p(\theta_1)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{y_1^2}-2\theta_1y_1+\theta_1^2\right)\right\}\exp\left\{-\frac{1}{2R_1}\left(\theta_1^2-2\theta_1a_1+\cancel{a_1^2}\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2\tau^2}\left(\cancel{(\theta_2-\alpha)^2}-2\beta\theta_1(\theta_2-\alpha)+\beta^2\theta_1^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta_1^2\left(\frac{1}{\sigma^2}+\frac{1}{R_1}+\frac{\beta^2}{\tau^2}\right)-2\theta_1\left(\frac{y_1}{\sigma^2}+\frac{a_1}{R_1}+\frac{\beta(\theta_2-\alpha)}{\tau^2}\right)\right]\right\}$$

$$\Rightarrow (\theta_1|\theta_2,y_1,\gamma)\sim N(m_1,C_1) \qquad C_1^{-1}=\sigma^{-2}+R_1^{-1}+\beta^2\tau^{-2}$$

$$m_1 = C_1\left(y_1/\sigma^2+a_1/R_1+\frac{\beta(\theta_2-\alpha)}{\tau^2}\right)$$

$\underline{t=n}$

$$p(\theta_n \mid \theta_{n-1}, y_n, \gamma) \propto p(y_n \mid \theta_n, \gamma)\, p(\theta_n \mid \theta_{n-1}, \gamma)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{y_n^2} - 2\theta_n y_n + \theta_n^2\right)\right\} \exp\left\{-\frac{1}{2\tau^2}\left(\theta_n^2 - 2\theta_n(\alpha+\beta\theta_{n-1}) + \cancel{(\alpha+\beta\theta_{n-1})^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta_n^2\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right) - 2\theta_n\left(\frac{y_n}{\sigma^2} + \frac{\alpha+\beta\theta_{n-1}}{\tau^2}\right)\right]\right\}$$

$$\Rightarrow (\theta_n \mid \theta_{n-1}, y_n, \gamma) \sim N(m_n, C_n)$$

$$C_n^{-1} = \sigma^{-2} + \tau^{-2}$$

$$m_n = C_n\left(\frac{y_n}{\sigma^2} + \frac{\alpha+\beta\theta_{n-1}}{\tau^2}\right)$$

Assuming $p(\alpha,\beta,\tau^2,\sigma^2) = p(\alpha)p(\beta)p(\tau^2)p(\sigma^2)$

where

$$\delta = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \sim N(d_0, D_0)$$

$$\tau^2 \sim IG(a_\tau, b_\tau) \quad \sigma^2 \sim IG(a_\sigma, b_\sigma)$$

we now derive the fixed parameters' full conditionals

$$p(\sigma^2 | y_{1:n}, \theta_{1:n}, \alpha, \beta, \tau^2) \propto (\sigma^2)^{-(a_\sigma+1)} e^{-\frac{b_\sigma}{\sigma^2}}$$

$$\times (\sigma^2)^{-(\frac{n}{2}+1)} e^{-\frac{\Sigma(y_t-\theta_t)^2/2}{\sigma^2}}$$

$$\sim IG\left(a_\sigma + n/2 \, ; \, b_\sigma + \sum_{t=1}^{n} (y_t-\theta_t)^2/2\right)$$

$$p(\tau^2 | y_{1:n}, \theta_{1:n}, \alpha, \beta, \sigma^2) \propto (\tau^2)^{-(a_\tau+1)} e^{-b_\tau/\tau^2}$$

$$\times (\tau^2)^{-(\frac{n-1}{2}+1)} e^{-\frac{\sum_{t=2}^{n}(\theta_t-\alpha-\beta\theta_{t-1})^2/2}{\tau^2}}$$

$$\sim IG\left(a_\tau + \frac{n-1}{2} \, ; \, b_\tau + \frac{\sum_{t=2}^{n}(\theta_t-\alpha-\beta\theta_{t-1})^2}{2}\right)$$

$$p(\delta | y_{1:n}, \theta_{1:n}, \tau^2, \sigma^2) \propto \exp\left\{-\frac{1}{2}(\delta-d_0)D_0^{-1}(\delta-d_0)\right\}$$

$\tilde{y} = (\theta_2, \dots, \theta_n)^T$

$$\times \exp\left\{-\frac{1}{2\tau^2}(\tilde{y}-X\delta)^T(\tilde{y}-X\delta)\right\}$$

$$X = \begin{bmatrix} 1 & \theta_1 \\ 1 & \theta_2 \\ \vdots & \vdots \\ 1 & \theta_{n-1} \end{bmatrix}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\delta^T(D_0^{-1} + X^TX/\tau^2) - 2\delta^T\left(D_0^{-1}d_0 + \frac{X^T\tilde{y}}{\tau^2}\right)\right]\right\}$$

$$\Rightarrow N\left[(D_0^{-1} + X^TX/\tau^2)^{-1}\left(D_0^{-1}d_0 + \frac{X^T\tilde{y}}{\tau^2}\right) ; \left(D_0^{-1} + \frac{X^TX}{\tau^2}\right)\right]$$

Local Level NDLM

$$y_t = \theta_t + \epsilon_t \qquad \epsilon_t \sim N(0, \sigma^2)$$
$$\theta_t = \theta_{t-1} + \omega_t \qquad \omega_t \sim N(0, \tau^2)$$
$$\theta_1 \sim N(a_1, R_1)$$

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For $t = 2, 3, \ldots, n-1$

$$p(\theta_t \mid \theta_{t-1}, \theta_{t+1}, y_t, \gamma) \propto p(y_t \mid \theta_t, \gamma)\, p(\theta_t \mid \theta_{t-1}, \gamma)\, p(\theta_{t+1} \mid \theta_t, \gamma)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{y_t^2} - 2y_t\theta_t + \theta_t^2\right)\right\} \exp\left\{-\frac{1}{2\tau^2}\left(\theta_t^2 - 2\theta_t\theta_{t-1} + \cancel{\theta_{t-1}^2}\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2\tau^2}\left(\cancel{\theta_{t+1}^2} - 2\theta_{t+1}\theta_t + \theta_t^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta_t^2\left(\frac{1}{\sigma^2} + \frac{2}{\tau^2}\right) - 2\theta_t\left(\frac{y_t}{\sigma^2} + \frac{\theta_{t-1} + \theta_{t+1}}{2}\right)\right]\right\}$$

$$\Rightarrow (\theta_t \mid \theta_{t-1}, \theta_{t+1}, y_t, \gamma) \sim N(m_t, C_t) \qquad C_t^{-1} = \sigma^{-2} + \tau^{-2}$$
$$m_t = C_t\left(y_t/\sigma^2 + \frac{\theta_{t-1} + \theta_{t+1}}{\tau^2}\right)$$

$t = 1$

$$p(\theta_1 \mid \theta_2, y_1, \gamma) \propto p(y_1 \mid \theta_1, \gamma)\, p(\theta_2 \mid \theta_1, \gamma)\, p(\theta_1)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left(y_1^2 - 2\theta_1 y_1 + \theta_1^2\right)\right\} \exp\left\{-\frac{1}{2\tau^2}\left(\theta_2^2 - 2\theta_1\theta_2 + \theta_1^2\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2R_1}\left(\theta_1^2 - 2\theta_1 a_1 + a_1^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta_1^2\left(\frac{1}{\sigma^2} + \frac{1}{\tau^2} + \frac{1}{R_1}\right) - 2\theta_1\left(\frac{y_1}{\sigma^2} + \frac{\theta_2}{\tau^2} + \frac{a_1}{R_1}\right)\right]\right\}$$

$$\Rightarrow (\theta_1 \mid \theta_2, y_1, \gamma) \sim N(m_1, C_1) \qquad C_1^{-1} = \sigma^{-2} + \tau^{-2} + R_1^{-1}$$
$$m_1 = C_1\left(y_1/\sigma^2 + \theta_2/\tau^2 + a_1/R_1\right)$$

$t = 2$ Similarly,

$$(\theta_n \mid \theta_{n-1}, y_n, \gamma) \sim N(m_n, C_n) \qquad C_n^{-1} = \sigma^{-2} + \tau^{-2}$$
$$m_n = C_n\left(\frac{y_n}{\sigma^2} + \frac{\theta_{n-1}}{\tau^2}\right)$$