PhD in Business Economics Advanced Bayesian Econometrics
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Please submit either your file (handwritten or typed) in PDF or HTML. The file must be a single PDF/HTML document for submission to me at `hedibertfl@insper.edu.br`. Students should follow the deadlines for submissions. This homework assignment should be done individually.

Nonlinear Gaussian regression

Suppose we observe data (x_i, y_i) for $i = 1, \ldots, n$ and consider the following nonlinear Gaussian model:

$$y_i|x_i, \beta, \gamma, \sigma^2 \sim N(\beta_0 + \beta_1 g(x_i, \gamma), \sigma^2),$$

where $\beta = (\beta_0, \beta_1)'$ and $g(x_i, \gamma) = x_i/(\gamma + x_i)$, for $\beta \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$.

Let us consider the prior for β, γ and σ^2 as follows:

$$\begin{aligned} p(\beta, \gamma, \sigma^2) &= p(\beta|\sigma^2)p(\gamma)p(\sigma^2) \\ \beta|\sigma^2 &\sim N(b_0, \sigma^2 B_0), \\ \gamma &\sim N(g_0, \tau_0^2), \\ \sigma^2 &\sim IG(\nu_0/2, \nu_0\sigma_0^2/2). \end{aligned}$$

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Answer the following questions.

(a) Show that $p(\beta|x, y, \sigma^2, \gamma)$ is a Gaussian distribution.

(b) Show that $p(\sigma^2|x, y, \beta, \gamma)$ is an Inverse-Gamma distribution.

(c) Show that $p(\sigma^2|x, y, \gamma)$ is also an Inverse-Gamma distribution.

Note 1: (c) is possible because we made the prior of β conditional on σ^2, $p(\beta|\sigma^2)$, which results in $p(\sigma^2|x, y, \gamma)$ also being an Inverse-Gamma distribution. What is the catch? Well, the catch is that `multiplying` (a) and (c) is `exactly` $p(\beta, \sigma^2|x, y, \gamma)$, while `iterating` between (a) and (b) is `approximately` $p(\beta, \sigma^2|x, y, \gamma)$, which is the standard Gibbs (or, more generally, MCMC) theorem (We will talk about these soon)

Note 2: It is not hard to see that $(\gamma|x, y, \beta, \sigma^2)$ comes from a distribution of no known form, since γ from the prior does not conjugate with γ from the likelihood function. Nonetheless, it can be point-wise evaluated up to a normalizing constant as

$$\begin{aligned} p(\gamma|x, y, \beta, \sigma^2) &\propto \exp\left\{-\frac{1}{2\tau_0^2}(\gamma^2 - 2\gamma m_0)\right\} \\ &\times \exp\left\{-\frac{1}{2\sigma^2}\left[\beta_1^2 \sum_{i=1}^n g^2(x_i, \gamma) - 2\sum_{i=1}^n (y_i - \beta_0) g(x_i, \gamma)\right]\right\}. \end{aligned}$$

Student's t as scale-mixture of Gaussians

Let

$$\theta|\mu, \sigma^2, \lambda \sim N(\mu, \lambda\sigma^2) \quad \text{and} \quad \lambda \sim IG(\nu/2, \nu/2),$$

for θ and μ in $\mathbb{R}$, σ^2, λ and ν in $\mathbb{R}^+$. Show that

$$\theta|\mu, \sigma^2 \sim t_\nu(\mu, \sigma^2).$$

In words, the Student's t distribution is a scale-mixture of normal distributions with an inverse-gamma as mixing distribution.

Try the following piece of code

```
set.seed(12355)
mu = 5
sigma = 2
nu = 4
theta = mu+sigma*rt(10000,df=nu)
lambda = 1/rgamma(10000,nu/2,nu/2)
theta1 = rnorm(1000,mu,sqrt(lambda)*sigma)
plot(density(theta),xlab=expression(theta),ylab="Student's t density",lwd=2,main="")
lines(density(theta1),col=2,lwd=2)
thetas = seq(min(theta,theta1),max(theta,theta1),length=1000)
lines(thetas,dt((thetas-mu)/sigma,df=nu)/sigma,col=4,lwd=2)
legend("topright",legend=c("Density","sampling directly",
"using scale-mixture argument"),col=c(4,1,2),lwd=2,bty="n")
```