HW1 - SOLUTION - Advanced Bayesian Econometrics

MODEL

$$y_1,\dots,y_m \text{ iid } \mathrm{Ber}(\theta) \Rightarrow L(\theta|y_{1:m}) = \theta^{S_n}(1-\theta)^{n-S_n} \qquad S_n = \sum_{i=1}^{n} y_i$$

\# of successes

$$\Downarrow$$

Looks like $\mathrm{Beta}(S_n+1, n-S_n+1)$

PRIOR

$$\theta \sim \pi\,\mathrm{Beta}(a,b) + (1-\pi)\,\mathrm{Beta}(c,d)$$

PREDICTIVE

$$p(y_{1:n}) = \int_0^1 \left(\pi \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} + \frac{(1-\pi)\,\theta^{c-1}(1-\theta)^{d-1}}{B(c,d)} \right) \theta^{S_n}(1-\theta)^{n-S_n}\, d\theta$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

$$a_1 = a+S_n, \quad b_1 = b+n-S_n, \quad c_1 = c+S_n, \quad d_1 = d+n-S_n$$

$$\Rightarrow p(y_{1:n}) = \frac{\pi}{B(a,b)} \int_0^1 \theta^{a+S_n-1}(1-\theta)^{b+n-S_n-1} d\theta + \frac{(1-\pi)}{B(c,d)} \int_0^1 \theta^{c+S_n-1}(1-\theta)^{d+n-S_n-1} d\theta$$

$$= \frac{\pi}{B(a,b)} B(a_1,b_1) \underbrace{\int_0^1 \frac{1}{B(a_1,b_1)} \theta^{a_1-1}(1-\theta)^{b_1-1} d\theta}_{=1} + \frac{(1-\pi)}{B(c,d)} B(c_1,d_1) \underbrace{\int_0^1 \frac{1}{B(c_1,d_1)} \theta^{c_1-1}(1-\theta)^{d_1-1} d\theta}_{=1}$$

$$\therefore\ p(y_{1:n}) = \pi \frac{B(a_1,b_1)}{B(a,b)} + (1-\pi) \frac{B(c_1,d_1)}{B(c,d)}$$

Consequently,

$$p(\theta|y_{1:n},a,b,c,d,\pi) = \frac{\pi \cdot \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} \theta^{S_n}(1-\theta)^{n-S_n} + (1-\pi) \frac{\theta^{c-1}(1-\theta)^{d-1}}{B(c,d)} \theta^{S_n}(1-\theta)^{n-S_n}}{\pi \frac{B(a_1,b_1)}{B(a,b)} + (1-\pi)\frac{B(c_1,d_1)}{B(c,d)}}$$

Since $\theta^{a+S_n-1}(1-\theta)^{b+n-S_n-1}$ is the Kernel of a $\mathrm{Beta}(a_1,b_1)$ and $\theta^{c+S_n-1}(1-\theta)^{d+n-S_n-1}$ is the Kernel of a $\mathrm{Beta}(c_1,d_1)$, it follows that

$$p(\theta|y_{1:n},a,b,c,d,\pi) = \frac{\frac{\pi B(a_1,b_1)}{B(a,b)} p_{Beta}(\theta|a_1,b_1) + \frac{(1-\pi)B(c_1,d_1)}{B(c,d)} p(\theta|c_1,d_1)}{\frac{\pi B(a_1,b_1)}{B(a,b)} + (1-\pi)\frac{B(c_1,d_1)}{B(c,d)}}$$

It is then easy to see that

$$p(\theta|y_{1:n},a,b,c,d,\pi) = \pi_1 P_{beta}(\theta|a_1,b_1) + (1-\pi_1) P_{beta}(\theta|c_1,d_1)$$

where

$$\pi_1 = \frac{1}{1 + \frac{(1-\pi)}{\pi} \cdot \frac{B(a,b)}{B(c,d)} \cdot \frac{B(c_1,d_1)}{B(a_1,b_1)}}$$