Exercises from Migon and Gamerman (Ch. 2)

Problem 10. Let $X\,|\theta,\mu \sim N(\theta,\sigma^2), \sigma^2$ known and $\theta\,|\mu \sim N(\mu,\tau^2)$, τ^2 known and $\mu \sim N(0,1)$. Obtain the following distributions:

1. $(\theta\,|x,\mu)$
2. $(\mu\,|x)$
3. $(\theta\,|x)$

Problem 11. Let $(X|\theta) \sim N(\theta,1)$ be observed. Suppose that your prior is such that θ is $N(\mu,1)$ or $N(-\mu,1)$ with equal probabilities. Write the prior distribution and find the posterior after observing $X = x$. Show that

$$E(\theta|x) = \frac{x}{2} + \frac{\mu}{2}\frac{1-\exp(-\mu x)}{1+\exp(-\mu x)}$$

and draw a graph of $E(\theta|x)$ as a function of x.

Problem 12. The standard Cauchy density function is

$$p(x|\theta) = (1/\pi)\{1/[1+(x-\theta)^2]\}$$

is similar to the $N(\theta,1)$ and can be used in its place in many applications. Get the modal equation (the first order condition to the maximum) of the posterior supposing that the prior is $p(\theta) = 1/\pi(1+\theta^2)$.

1. Solve it for $x = 0$ and $x = 3$.
2. Compare with the results obtained assuming that $(x|\theta) \sim N(\theta,1)$ and $\theta \sim N(0,1)$.

Problem 17. Let $X = (X_1,\ldots,X_n)$ be a random sample from the $U(\theta_1,\theta_2)$, that is,

$$p(x\,|\theta_1,\theta_2) = \frac{1}{\theta_2-\theta_1},\ \theta_1 \leq x \leq \theta_2.$$

Let $T(X) = (X_{(1)}, X_{(n)})$, obtain its joint distribution and show that it is a sufficient statistic for $\theta = (\theta_1,\theta_2)$.

Problem 22. Let $X_1, \ldots, X_n$ be a random sample from the $N(\mu, \sigma^2)$,with σ^2 unknown. Show, using the classical definition, that the sample mean $\overline{X}$ is a sufficient statistic for μ. *Hint:* It is enough to show that $E(X|\overline{X})$ and $V(X|\overline{X})$ are not functions of μ. Why?

Problem 30. Let X and Y be independent random variables Poisson distributed with parameters θ and ϕ, respectively, and suppose that the prior distribution is

$$p(\theta, \phi) = p(\theta)p(\phi) \propto k.$$

Let $\psi = \theta/(\theta + \phi)$ and $\xi = \theta + \phi$ be a parametric transformation of (θ, ϕ).

1. Obtain the prior for (ψ, ξ).
2. Show that $\psi|x, y \sim Beta(x + 1, y + 1)$ and $\xi|x, y \sim G(x + y + 2, 1)$ are independent.
3. Show that the conditional distribution X given $X + Y$ depends only on ψ, that is $p(x|x + y, \psi, \xi) = p(x|x + y, \psi)$ and that the distribution of $X + Y$ depends only on ξ.
4. Show that $X + Y$ is a sufficient statistics for ξ, given $X + Y$, X is a sufficient statistic for ψ and that $(X, X + Y)$ is a sufficient statistic for (ψ, ξ).
5. Obtain the marginal likelihood of ψ and ξ.
6. To make inference for ξ a statistician decides to use the fact presented in item (d). Show that the posterior is identical to that obtained em (b). Does it mean that $X + Y$ does not contain information about ψ?

Problem 33. Let (X_1, X_2, X_3) be a random vector with distribution trinomial with parameter $\theta = (\theta_1, \theta_2, \theta_3)$ where $\theta_3 = 1 - \theta_1 - \theta_2$ and assume that the prior for θ is constant.

1. Define $\lambda = \theta_1/(\theta_1 + \theta_2)$ and $\psi = \theta_1 + \theta_2$ and obtain their priors.
2. Obtain the marginal likelihood of ψ.
3. Show that $X_1 + X_2$ is a sufficient statistics for ψ.