1. $w|\nu \sim IG(\frac{\nu}{2}, \frac{\nu}{2})$ & $z|w \sim N(0, w)$

$$p(z|\nu) = \int_0^\infty p(z|w)p(w|\nu)dw$$

$$= \int_0^\infty (2\pi w)^{-1/2} e^{-z^2/2w} \frac{(\nu/2)^{(\nu/2)}}{\Gamma(\nu/2)} w^{-(\frac{\nu}{2}+1)} e^{-\frac{\nu/2}{w}} dw$$

$$= \frac{2\pi (\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \int_0^\infty \underbrace{w^{-(\frac{\nu+1}{2}+1)} e^{-\frac{(z^2+\nu)/2}{w}}}_{\text{Kernel of } IG(\frac{\nu+1}{2}, \frac{z^2+\nu}{2})} dw$$

$$= \frac{2\pi (\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \frac{\Gamma(\frac{\nu+1}{2})}{(\frac{z^2+\nu}{2})^{\frac{\nu+1}{2}}} \propto \underbrace{(1+\nu^{-1} z^2)^{-\frac{\nu+1}{2}}}_{\text{Kernel of } t_\nu(0,1)}$$

$\therefore$ $z|\nu \sim t_\nu(0,1)$ #

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2. $$P(\mu \mid y_{1:n}, \sigma^2, \lambda_{1:n}) \propto p(\mu) \prod_{i=1}^{n} p_N(y_i \mid \mu, \lambda_i \sigma^2)$$

$$\propto \exp\left\{-\frac{1}{2}\left(\mu^2/c_0 - 2\mu m_0/c_0\right)\right\}$$

$$\times \underbrace{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2/\lambda_i\sigma^2\right\}}_{\mu^2 \frac{\sum 1/\lambda_i}{\sigma^2} - 2\mu \frac{\sum y_i/\lambda_i}{\sigma^2} + \text{constant}}$$

$$\therefore \mu \mid y_{1:n}, \sigma^2, \lambda_{1:n} \sim N\left[\left(\frac{1}{c_0} + \frac{1}{\sigma^2}\sum_{i=1}^{n} 1/\lambda_i\right)^{-1}\left(\frac{m_0}{c_0} + \frac{\sum_{i=1}^{n} y_i/\lambda_i}{\sigma^2}\right);\ \left(\frac{1}{c_0} + \frac{1}{\sigma^2}\sum_{i=1}^{n} 1/\lambda_i\right)^{-1}\right]$$

3. $$p(\sigma^2 \mid y_{1:n}, \mu, \lambda_{1:n}) \propto p(\sigma^2) \prod_{i=1}^{n} p_N(y_i \mid \mu, \lambda_i \sigma^2)$$

$$\propto (\sigma^2)^{-(a_0+1)} e^{-b_0/\sigma^2} (\sigma^2)^{-\frac{n}{2}} e^{-\frac{\left(\sum_{i=1}^{n}(y_i-\mu)^2/\lambda_i\right)/2}{\sigma^2}}$$

$$\therefore (\sigma^2 \mid y_{1:n}, \mu, \lambda_{1:n}) \sim IG\left(a_0 + \frac{n}{2};\ b_0 + \frac{\sum_{i=1}^{n}(y_i-\mu)^2/\lambda_i}{2}\right)$$

4. $p(\lambda_i | \lambda_{-i}, y_{1:n}, \mu, \sigma^2) = p(\lambda_i | y_i, \mu, \sigma^2) p(\lambda_i)$

$$\propto \lambda_i^{-\left(\frac{\nu}{2}+1\right)} e^{-\frac{\nu/2}{\lambda_i}}$$

$$\times \lambda_i^{-1/2} e^{-\frac{(y_i-\mu)^2/2\sigma^2}{\lambda_i}}$$

$$\propto \lambda_i^{\left(\frac{\nu+1}{2}+1\right)} e^{-\frac{\frac{\nu+(y_i-\mu)^2/\sigma^2}{2}}{\lambda_i}}$$

$$\therefore \lambda_i | \lambda_{-i}, y_{1:n}, \mu, \sigma^2 \sim IG\left(\frac{\nu+1}{2}, \frac{\nu+(y_i-\mu)^2/\sigma^2}{2}\right).$$

5. See R code

6. See R code

7 & 8 See R Code from Professor's page about Bayesian linear regression with Gaussian errors.