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$\{y_1, y_2, \dots y_n\} \equiv$ DATA (COUNTS)

Model 0: $y_i|\lambda \overset{iid}{\sim} Poi(\lambda) \quad \lambda > 0 \quad y_i \in \{0,1,2,\dots\}$

$\lambda \sim G(\alpha_0, \beta_0) \quad \alpha_0, \beta_0 > 0$

Likelihood + prior $\Rightarrow$ $p(y|\lambda) = \dfrac{\lambda^{\sum_{i=1}^{n} y_i} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!} \Rightarrow L(\lambda|y) \propto \lambda^{S_n} e^{-n\lambda}$

$S_n = \sum_{i=1}^{n} y_i$

$p(\lambda) = \dfrac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \lambda^{\alpha_0 - 1} e^{-\beta_0 \lambda} \propto \lambda^{\alpha_0 - 1} e^{-\beta_0 \lambda} \qquad E(\lambda) = \dfrac{\alpha_0}{\beta_0}$

posterior $\Rightarrow p(\lambda|y) \propto \lambda^{(\alpha_0 + S_n) - 1} e^{-(\beta_0 + n)\lambda} \equiv G(\underbrace{\alpha_0 + S_n}_{\alpha_1}; \underbrace{\beta_0 + n}_{\beta_1})$

$E(\lambda|y) = \dfrac{\alpha_0 + S_n}{\beta_0 + n}$

prior predictive $\Rightarrow p(S_n) = \dfrac{1}{\prod_{i=1}^{n} y_i!} \dfrac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \displaystyle\int_0^{\infty} \lambda^{(\alpha_0 + S_n) - 1} e^{-(\beta_0 + n)\lambda} d\lambda$

$\Rightarrow \boxed{p(S_n) = \dfrac{1}{\prod_{i=1}^{n} y_i!} \times \dfrac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \times \dfrac{\Gamma(\alpha_0 + S_n)}{(\beta_0 + n)^{\alpha_0 + S_n}} \qquad S_n = 0, 1, 2, \dots}$

All derivations ARE in closed form!

Model 1: $y_i|\theta \overset{iid}{\sim} Poi(\theta) \qquad \theta > 0 \qquad y_i \in \{0,1,2,\ldots\}$

$$\log(\theta) \sim N(\mu_0, \sigma_0^2) \Rightarrow p(\theta) = \frac{1}{\theta\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{(\ln\theta - \mu_0)^2}{2\sigma_0^2}\right\}$$

$$\Rightarrow \theta \sim LN(\mu_0, \sigma_0^2)$$

$$E(\theta) = \exp(\mu_0 + \sigma_0^2/2)$$

$$\text{Median}(\theta) = \exp(\mu_0)$$

$$\text{Mode}(\theta) = \exp(\mu_0 - \sigma_0^2)$$

$$\text{Var}(\theta) = (\exp(\sigma_0^2) - 1)\exp(2\mu_0 + \sigma_0^2)$$

Matching priors:

$$E(\lambda|M_0) = \frac{\alpha_0}{\beta_0} \qquad E(\theta/M_1) = e^{\mu_0 + \sigma_0^2/2}$$

$$V(\lambda/M_0) = \frac{\alpha_0}{\beta_0^2} \qquad V(\theta/M_1) = (e^{\sigma_0^2} - 1)e^{2\mu_0 + \sigma_0^2}$$

Posterior: $$p(\theta|y) \propto \theta^{S_n} e^{-n\theta}\, \theta^{-1} e^{-\frac{(\ln\theta - \mu_0)^2}{2\sigma_0^2}} \qquad \theta > 0$$

$\Rightarrow$ no known closed form distribution!

Prior Predictive: $$p(y) = \int_0^\infty \left(\frac{1}{\prod_{i=1}^n y_i!} \cdot \frac{1}{\sqrt{2\pi\sigma_0^2}}\right) \underbrace{\theta^{S_n - 1} e^{-n\theta} e^{-\frac{(\ln\theta - \mu_0)^2}{2\sigma_0^2}}}_{g(\theta)} d\theta$$

$$\Rightarrow p(y) = \left(\frac{1}{\prod_{i=1}^n y_i! \sqrt{2\pi\sigma_0^2}}\right) \int_0^\infty g(\theta)\, d\theta.$$

Bayes factor

$$B_{01} = \frac{\frac{1}{\prod_{i=1}^{n} y_i!} \times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \times \frac{\Gamma(\alpha_0 + S_n)}{(\beta_0 + n)^{\alpha_0 + S_n}}}{\frac{1}{\prod_{i=1}^{n} y_i!} \times \frac{1}{\sqrt{2\pi\sigma_0^2}} \times \int_0^\infty g(\theta)\,d\theta}$$

$$\Rightarrow \boxed{B_{01} = \frac{\beta_0^{\alpha_0}(2\pi\sigma_0^2)^{1/2}}{\Gamma(\alpha_0)} \times \frac{\Gamma(\alpha_0 + S_n)}{(\beta_0 + n)^{\alpha_0 + S_n}} \times \frac{1}{\int_0^\infty g(\theta)\,d\theta}}$$

Computational Problem : $I = \int_0^\infty g(\theta)\,d\theta$ (this is a scalar!)

Monte Carlo Integration via importance function

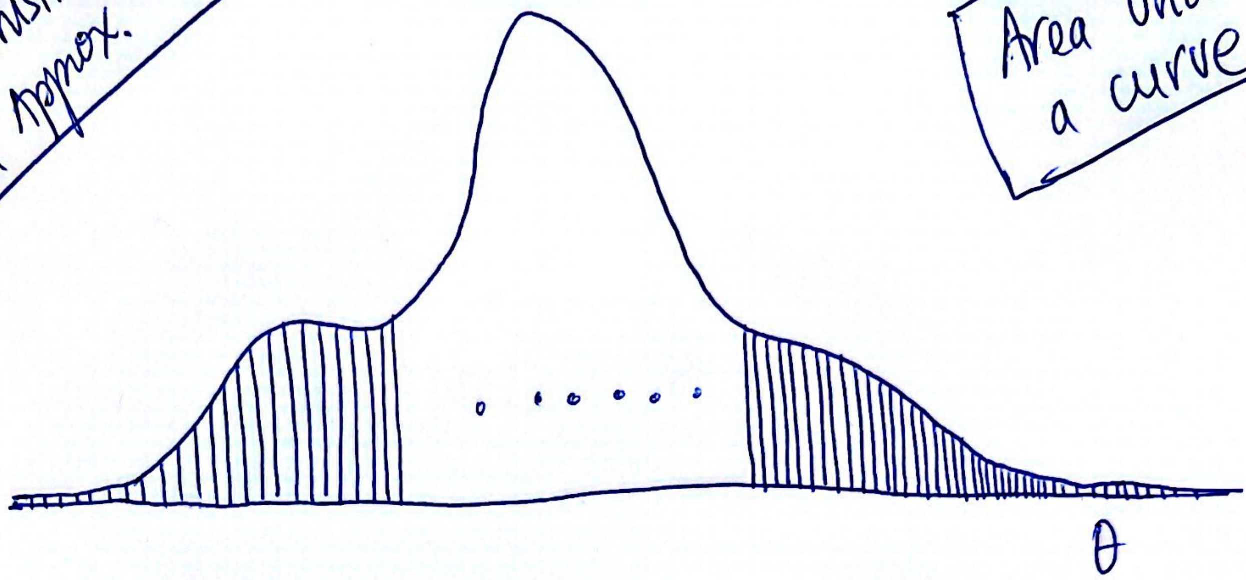
$$\hat{I}_{MCIF} = \frac{1}{M} \sum_{j=1}^{M} \frac{g(\theta^{(j)})}{f(\theta^{(j)})} \xrightarrow{M \to \infty} \int_0^\infty \left\{ \frac{g(\theta)}{f(\theta)} \right\} f(\theta)\,d\theta$$

where $\{\theta^{(i)}\}_{i=1}^{M} \overset{iid}{\sim} f(\theta)$

KEY ISSUE: How to pick $f(\theta)$?

Area under a curve

Deterministic Grid Approx.



Stochastic/MC Approximation