Bayesian Learning - MPE 2021-2

Hedibert Lopes - Insper

Question 0 - April 27th 2021

Triple head

In your pocket you have three coins:
one with 50% probability of head (coin 1),
one with 90% probability of head (coin 2), and
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and that

$$\begin{aligned} Pr(\text{three heads} \mid \text{coin } 1) &= (5/10)^3 = 0.125, \\ Pr(\text{three heads} \mid \text{coin } 2) &= (9/10)^3 = 0.729, \\ Pr(\text{three heads} \mid \text{coin } 3) &= (10/10)^3 = 1.000. \end{aligned}$$

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$$Pr(\text{coin 1} \mid \text{three heads})$$

Bayes theorem

For $Pr(B) > 0$,

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{Pr(B|A)Pr(A)}{Pr(B)} = \left(\frac{Pr(B|A)}{Pr(B)}\right) Pr(A)$$

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Easy to see that

$$\begin{aligned} Pr(\text{coin 2} \mid \text{three heads}) &= \frac{(0.729)(1/3)}{Pr(\text{three heads})} \\ Pr(\text{coin 3} \mid \text{three heads}) &= \frac{(1.000)(1/3)}{Pr(\text{three heads})}. \end{aligned}$$

Computing Pr(three heads)

For A, B, C such that $Pr(B \cup C) = 1$ and $Pr(B \cap C) = 0$,

$$\begin{aligned} Pr(A) &= Pr(A \cap B) + Pr(A \cap C) \\ &= Pr(A|B)Pr(B) + Pr(A|C)Pr(C). \end{aligned}$$

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To compute Pr(three heads), we need to figure out the "weighted average" of the outcome (three heads) based on ALL possible scenarios, i.e. coin 1 or coin 2 or coin 3:

$$\begin{aligned} Pr(\text{three heads}) &= Pr(\text{three heads} \mid \text{coin 1})Pr(\text{coin 1}) \\ &+ Pr(\text{three heads} \mid \text{coin 2})Pr(\text{coin 2}) \\ &+ Pr(\text{three heads} \mid \text{coin 3})Pr(\text{coin 3}) \\ &= (0.125)(1/3) + (0.729)(1/3) + (1.000)(1/3) \\ &= 1.854/3 = 0.618. \end{aligned}$$

Back to what we want

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\begin{aligned}
Pr(\text{coin 1} \mid \text{three heads}) &= \frac{Pr(\text{three heads} \mid \text{coin 1})Pr(\text{coin 1})}{Pr(\text{three heads})} \\
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&= \frac{0.04166667}{0.618} = 0.0674.
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Pr(\text{coin 2} \mid \text{three heads}) &= 0.3932 \\
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Question: What if a fourth flip turned out head as well?