$i = 1, 2, \dots, n \qquad \text{Data} \equiv \{(x_i, y_i) : i = 1, \dots, n\}$

$$y_i = \beta x_i + \sigma \varepsilon_i \qquad \varepsilon_i \overset{iid}{\sim} N(0,1)$$

$$p(\beta) \equiv N(b_0, \tau_0^2) \qquad \beta \in \mathbb{R}$$

$$C^+(0,1) : \quad p(\sigma^2) = \frac{2}{\pi(1+\sigma^2)} \qquad \sigma^2 > 0$$

$$p(\beta, \sigma^2 | y_{1:n}) \propto \frac{1}{\pi(1+\sigma^2)} \exp\left\{-\frac{1}{2\tau_0^2}(\beta - b_0)^2\right\}$$
$$\times (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2}$$

Full Conditionals

$$p(\beta|\sigma^2, y_{1:n}) \propto \exp\left\{-\frac{1}{2}\left[\beta^2\left(\frac{1}{\tau_0^2} + \frac{\sum x_i^2}{\sigma^2}\right) - 2\beta\left(\frac{b_0}{\tau_0^2} + \frac{\sum x_i y_i}{\sigma^2}\right)\right]\right\}$$

$$\Rightarrow (\beta|\sigma^2, y_{1:n}) \sim N\left[\left(\frac{1}{\tau_0^2} + \frac{\sum x_i^2}{\sigma^2}\right)^{-1}\left(\frac{b_0}{\tau_0^2} + \frac{\sum x_i y_i}{\sigma^2}\right); \left(\frac{1}{\tau_0^2} + \frac{\sum x_i^2}{\sigma^2}\right)^{-1}\right]$$

Know distribution $\Rightarrow$ easy to draw from

$$p(\sigma^2|\beta, y_{1:n}) \propto \frac{1}{(1+\sigma^2)(\sigma^2)^{n/2}} \exp\left\{-\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - x_i\beta)^2/2\right\}$$

unknow distribution $\Rightarrow$ Maybe use a Metropolis-Hastings step

Notice that $p(\sigma^2|\beta, y_{1:n}) \propto \frac{1}{(1+\sigma^2)} \cdot p_{IG}\left(\sigma^2 \,\middle|\, \frac{n-2}{2}; \frac{\sum(y_i - x_i\beta)^2}{2}\right)$

Therefore, a "natural candidate" for proposal is $q(\sigma^2|\beta) \equiv IG\left(\sigma^2 \,\middle|\, \frac{n-2}{2}; \frac{\sum_{i=1}^{n}(y_i - x_i\beta)^2}{2}\right)$

MCMC scheme

0) Start with $\sigma^{2(0)} = 1$ (or any other value in $\mathbb{R}^+$)

For $i=0,\ldots,M$, repeat A) & B)

A) Sample $\beta^{(i+1)} \sim N\left(\left(\frac{1}{c_0^2} + \frac{\sum x_i^2}{\sigma^{2(i)}}\right)^{-1}\left(\frac{b_0}{c_0^2} + \frac{\sum x_i y_i}{\sigma^{2(i)}}\right); \left(\frac{1}{c_0^2} + \frac{\sum x_i^2}{\sigma^{2(i)}}\right)^{-1}\right).$

GIBBS-STEP

RW-MH STEP

B) Sample $(\sigma^2)^* \sim IG\left(\frac{n-2}{2}, \frac{\sum_{i=1}^{n}(y_i - x_i\beta^{(i+1)})^2}{2}\right)$

Accept $(\sigma^2)^*$ w.p. $\alpha = \min\{1, A\}$

$$A = \frac{\frac{1}{1+(\sigma^2)^*}}{\frac{1}{1+\sigma^{2(i)}}} = \frac{1+\sigma^{2(i)}}{1+(\sigma^2)^*}.$$

$$(\sigma^2)^{(i+1)} = \begin{cases} (\sigma^2)^{(i)} & \text{w.p. } (1-\alpha) \\ (\sigma^2)^* & \text{w.p. } \alpha \end{cases}$$