(a) Conditionally on x, y, σ^2 and γ, the vector $\beta = (\beta_0, \beta_1)^T$ appears in the likelihood in the same way it appear in the prior. More specifically,

$$p(\beta | x, y, \sigma^2, \gamma) \propto \exp\left\{-\frac{1}{2\sigma^2}(y - \tilde{X}\beta)^T(y - X\beta)\right\} \qquad \underset{m\times 2}{\tilde{X}} = \begin{bmatrix} 1 & g(x_1,\gamma) \\ 1 & g(x_2,\gamma) \\ \vdots & \vdots \\ 1 & g(x_m,\gamma) \end{bmatrix}$$

$$\times \exp\left\{-\frac{1}{2\sigma^2}(\beta - b_0) B_0^{-1} (\beta - b_0)\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left[\beta^T(\tilde{X}^T\tilde{X} + B_0^{-1})\beta - 2\beta(\tilde{X}^T y + B_0^{-1} b_0)\right]\right\}$$

$$\Rightarrow (\beta | x, y, \sigma^2, \gamma) \sim N\left[\underbrace{(\tilde{X}^T\tilde{X} + B_0^{-1})^{-1}(\tilde{X}^T y + B_0^{-1} b_0)}_{b_1} ; \sigma^2 \underbrace{(\tilde{X}^T\tilde{X} + B_0^{-1})^{-1}}_{B_1}\right]$$

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(b) $$p(\sigma^2 | x, y, \beta, \gamma) \propto \exp\left\{-\frac{(y - \tilde{X}\beta)^T(y - \tilde{X}\beta)/2}{\sigma^2}\right\}(\sigma^2)^{-m/2}$$

$$\times (\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)} e^{-\frac{\nu_0\sigma_0^2/2}{\sigma^2}}$$

$$\propto \exp\left\{-\frac{[\nu_0\sigma_0^2 + (y - \tilde{X}\beta)^T(y - \tilde{X}\beta)]/2}{\sigma^2}\right\}(\sigma^2)^{-\left(\frac{\nu_0+m}{2}+1\right)}$$

$$\Rightarrow (\sigma^2 | x, y, \beta, \gamma) \sim IG\left(\frac{\nu_0 + m}{2} ; \frac{\nu_0\sigma_0^2 + (y - \tilde{X}\beta)^T(y - X\beta)}{2}\right)$$

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(c) Here, you will need to use all your attention to details. I will guide you through the derivations.

$$P(\sigma^2|x,y,\gamma) = \int_{\mathbb{R}^p} P(\sigma^2,\beta|x,y,\gamma)\,d\beta = \int_{\mathbb{R}^p} P(y|x,\gamma,\beta,\sigma^2)\,P(\beta,\sigma^2)\,d\beta$$

$p=2$
(β_0,β_1)

$$= \int_{\mathbb{R}^p} (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}(y-\tilde{X}\beta)^T(y-\tilde{X}\beta)} (2\pi)^{-\frac{1}{2}} |B_0|^{-1/2} e^{-\frac{1}{2\sigma^2}(\beta-b_0)^T B_0^{-1}(\beta-b_0)} \times$$

$$\frac{(\sigma_0^2\nu_0/2)^{\nu_0/2}}{\Gamma(\nu_0/2)} (\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)} e^{-\frac{\nu_0\sigma_0^2/2}{\sigma^2}}\,d\beta$$

$$\propto (\sigma^2)^{-\frac{n}{2}} (\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)} e^{-\frac{\nu_0\sigma_0^2/2}{\sigma^2}} \int_{\mathbb{R}^p} \exp\left\{-\frac{[y^Ty - 2\beta^T\tilde{X}^Ty + \beta^T\tilde{X}^T\tilde{X}\beta + b_0^TB_0^{-1}b_0 - 2\beta^TB_0^{-1}b_0 + \beta^TB_0^{-1}\beta]}{2\sigma^2}\right.\,d\beta$$

$$\propto (\sigma^2)^{-\frac{n}{2}} (\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)} e^{-\frac{\nu_0\sigma_0^2/2}{\sigma^2}} e^{-\frac{[y^Ty + b_0^TB_0^{-1}b_0]/2}{\sigma^2}} \times$$

$$\int_{\mathbb{R}^p} \exp\left\{-\frac{\beta^T\overbrace{(B_0^{-1}+\tilde{X}^T\tilde{X})}^{B_1^{-1}}\beta - 2\beta^T(B_0^{-1}b_0 + \tilde{X}^Ty)}{\sigma^2}\right\}d\beta$$

since $B_1^{-1}b_1 = B_0^{-1}b_0 + \tilde{X}^Ty$

$$\propto (\sigma^2)^{-\left(\frac{\nu_0+n}{2}+1\right)} e^{-\frac{[\nu_0\sigma_0^2 + y^Ty + b_0^TB_0^{-1}b_0]/2}{\sigma^2}} \int_{\mathbb{R}^p} e^{-\frac{(\beta^TB_1^{-1}\beta - 2\beta^TB_1^{-1}b \pm b_1^TB_1^{-1}b_1)/2}{\sigma^2}}\,d\beta$$

$$\propto \underbrace{(\sigma^2)^{-\left(\frac{\nu_0+n}{2}+1\right)} e^{-\frac{[\nu_0\sigma_0^2 + y^Ty + b_0^TB_0^{-1}b_0 - b_1^TB_1^{-1}b_1]/2}{\sigma^2}}}_{IG\left(\frac{\nu_0+n}{2};\ \frac{\nu_0\sigma_0^2 + (y-\tilde{X}b_1)^Ty + (b_0-b_1)^TB_0^{-1}b_0}{2}\right)} \underbrace{\int_{\mathbb{R}} e^{-\frac{(\beta-b_1)^TB_1^{-1}(\beta-b_1)}{2\sigma^2}}\,d\beta}_{(2\pi\sigma^2)^{-\frac{p}{2}}|B_1|^{-1/2} \propto 1}$$

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