A

Modeling iid count data with Binomial (M_0) & with Poisson (M_1) models

Insper EDUCAÇÃO EXECUTIVA

Data: $\{y_1, y_2, \dots, y_n\}$ — counts

Model M_0: Binomial(n_0, θ) $\quad \theta \in (0,1)$, n_0 known

Model M_1: Poisson(λ) $\quad \lambda \geq 0$

Priors: $\theta \sim \text{Beta}(a_0, b_0)$; $\lambda \sim G(c_0, d_0)$

a_0, b_0, c_0, d_0 known hyperparameters (all >0)

$\Rightarrow$ Show that all quantities (predictives, posteriors, ect) can be computed in closed form.

a) $(\theta | y_1 \dots y_n, M_0) \sim \text{Beta}(a_1, b_1)$ $\qquad a_1 = a_0 + \sum_{i=1}^{n} y_i$, $\quad b_1 = b_0 + n n_0 - \sum_{i=1}^{n} y_i$

b) $(\lambda | y_1, \dots, y_n, M_1) \sim \text{Gamma}(c_1, d_1)$ $\qquad c_1 = c_0 + \sum_{i=1}^{n} y_i$, $\quad d_1 = d_0 + n$

c) $$\Pr(y_1, \dots y_n | M_0) = \frac{\Gamma(a_0 + b_0)}{\Gamma(a_0)\Gamma(b_0)} \prod_{i=1}^{n} \binom{n_0}{y_i} \frac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1 + b_1)}$$

d) $$\Pr(y_1, \dots, y_n | M_1) = \frac{d_0^{c_0}}{\Gamma(c_0)} \prod_{i=1}^{n} \left(\frac{1}{y_i!}\right) \frac{\Gamma(c_1)}{d_1^{c_1}}$$

e) $$\Pr(y_{n+1} | y_1, \dots, y_n, M_0) = \binom{n_0}{y_{n+1}} \frac{\Gamma(a_1 + b_1)}{\Gamma(a_1)\Gamma(b_1)} \frac{\Gamma(a_1 + y_{n+1})\Gamma(b_1 + n_0 - y_{n+1})}{\Gamma(a_1 + b_1 + n_0)}$$

$y_{n+1} = 0, 1, \dots, n_0$

e) $$\Pr(y_{n+1} | y_1, \dots, y_n, M_1) = \frac{1}{y_{n+1}!} \cdot \frac{d_1^{c_1}}{\Gamma(c_1)} \cdot \frac{\Gamma(c_1 + y_{n+1})}{(d_1 + 1)^{c_1 + y_{n+1}}} ; \; y_{n+1} = 0, 1, 2, \dots$$

(B)

$\theta \sim Beta(a_0, b_0)$ $y_1 \dots y_n$ iid $Bin(n_0, \theta)$

$\lambda \sim G(c_0, d_0)$ $y_1 \dots y_n$ iid $Poi(\lambda)$

$$p(\theta|y) \propto \theta^{a_0-1}(1-\theta)^{b_0-1} \times \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{n_0-y_i}$$

$$\propto \theta^{\left(a_0+\sum_{i=1}^{n} y_i\right)-1}(1-\theta)^{\left(b_0+n_0 \cdot n-\sum_{i=1}^{n} y_i\right)-1}$$

$$\sim Beta\Big(\underbrace{a_0+\sum y_i}_{a_1}, \underbrace{b_0+n n_0-\sum_{i=1}^{n} y_i}_{b_1}\Big)$$

$$E(\theta|y) = \frac{a_0+\sum y_i}{a_0+b_0+n n_0}$$

$$V(\theta|y) = \frac{(a_0+\sum y_i)(b_0+n n_0-\sum y_i)}{(a_0+b_0+n n_0)^2(a_0+b_0+n n_0)}$$

$$p(\lambda|y) \propto \lambda^{c_0-1} e^{-d_0\lambda} \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

$$\propto \lambda^{(c_0+\sum y_i)-1} e^{-(d_0+n)\lambda}$$

$$\sim G\Big(\underbrace{c_0+\sum_{i=1}^{n} y_i}_{c_1}, \underbrace{d_0+n}_{d_1}\Big)$$

$$E(\lambda|y) = \frac{c_0+\sum_{i=1}^{n} y_i}{d_0+n}$$

$$SD(\lambda|y) = \frac{\sqrt{c_0+\sum_{i=1}^{n} y_i}}{d_0+n}$$

(c)

$(\theta|y) \sim \text{Beta}(a_1, b_1)$

$(\lambda|y) \sim G(c_1, d_1)$

$a_1 = a_0 + \sum y_i$

$b_1 = b_0 + n n_0 - \sum y_i$

$c_1 = c_0 + \sum y_i$

$d_1 = d_0 + n$

$$P(y_1, \dots, y_n | M_0) = \int_0^1 \frac{\Gamma(a_0+b_0)}{\Gamma(a_0)\Gamma(b_0)} \theta^{a_0-1} (1-\theta)^{b_0-1} \times \prod_{i=1}^{n} \binom{n_0}{y_i} \theta^{y_i} (1-\theta)^{n_0 - y_i} d\theta$$

$$= \frac{\Gamma(a_0+b_0)}{\Gamma(a_0)\Gamma(b_0)} \prod_{i=1}^{n} \binom{n_0}{y_i} \underbrace{\int_0^1 \theta^{a_1-1} (1-\theta)^{b_1-1} d\theta}_{\frac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1+b_1)}}$$

$$P(y_1, \dots, y_n | M_1) = \int_0^\infty \frac{d_0^{c_0}}{\Gamma(c_0)} \lambda^{c_0-1} e^{-d_0\lambda} \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} d\lambda$$

$$= \frac{d_0^{c_0}}{\Gamma(c_0)} \cdot \prod_{i=1}^{n} \left(\frac{1}{y_i!}\right) \underbrace{\int_0^\infty \lambda^{c_1-1} e^{-d_1\lambda} d\lambda}_{\frac{\Gamma(c_1)}{d_1^{c_1}}}$$

D

$$P_r(y_{n+1}|y_1 \dots y_n, M_0) = \int_0^1 \binom{n_0}{y_{n+1}} \theta^{y_{n+1}} (1-\theta)^{n_0 - y_{n+1}} \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} \theta^{a_1-1} (1-\theta)^{b_1-1} d\theta$$

$$= \binom{n_0}{y_{n+1}} \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} \underbrace{\int \theta^{(a_1+y_{n+1})-1} (1-\theta)^{(b_1+n_0-y_{n+1})-1} d\theta}_{\frac{\Gamma(a_1+y_{n+1})\Gamma(b_1+n_0-y_{n+1})}{\Gamma(a_1+b_1+n_0)}}$$

$y_{n+1} = 0, 1, \dots, n_0$

$$P_r(y_{n+1}|y_1, \dots, y_n, M_1) = \int_0^\infty \frac{\lambda^{y_{n+1}} e^{-\lambda}}{y_{n+1}!} \frac{d_1^{c_1}}{\Gamma(c_1)} \cdot \lambda^{c_1-1} e^{-d_1\lambda} d\lambda$$

$$= \frac{1}{y_{n+1}!} \frac{d_1^{c_1}}{\Gamma(c_1)} \underbrace{\int \lambda^{(c_1+y_{n+1})-1} e^{-(d_1+1)\lambda} d\lambda}_{\frac{\Gamma(c_1+y_{n+1})}{(d_1+1)^{c_1+y_{n+1}}}}$$

$y_{n+1} = 0, 1, \dots,$