Bayesian Ingredients:
A brief introduction

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Outline

Bayesian paradigm: an overview

Example 1: Is Diego ill?
  Adding some modeling
  $X = 1$ is observed
  Bayesian learning

Example 2: Gaussian measurement error
  Large and small prior experience
  Bayesian computation: predictive
  Bayesian computation: posterior
  A small computational problem

Monte Carlo: a toy example
Bayesian paradigm: an overview

- Combination of different sources/levels of information
- Sequential update of beliefs
- A single, coherent framework for
  - Statistical inference/learning
  - Model comparison/selection/criticism
  - Predictive analysis and decision making
- Drawback: Computationally challenging
Example 1: Is Diego ill?

- Diego claims some discomfort and goes to his doctor.

- His doctor believes he might be ill (he may have the flu).

- $\theta = 1$: Diego is ill.
- $\theta = 0$: Diego is not ill.

- $\theta$ is the “state of nature” or “proposition”
Adding some modeling

The doctor can take a binary and imperfect “test” $X$ in order to learn about $\theta$:

$$\begin{cases} 
P(X = 1|\theta = 0) = 0.40, & \text{false positive} \\
P(X = 0|\theta = 1) = 0.05, & \text{false negative} 
\end{cases}$$

These numbers might be based, say, on observed frequencies over the years and over several hospital in a given region.
$X = 1$ is observed

**Data collection**

The doctor performs the test and observes $X = 1$. 

Maximum likelihood argument

$X = 1$ is more likely from a ill patient than from a healthy one

$P(X = 1 | \theta = 1) \approx 0.95$

$P(X = 1 | \theta = 0) = 0.40 = 0.375$

The maximum likelihood estimator of $\theta$ is $\hat{\theta}_{MLE} = 1$. 
$X = 1$ is observed

**Data collection**
The doctor performs the test and observes $X = 1$.

**Decision making**
How should the doctor proceed?
X = 1 is observed

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\[X \stackrel{\text{def}}{=} 1\] is observed
Bayesian learning

Suppose the doctor claims that

\[ P(\theta = 1) = 0.70 \]
Bayesian learning

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This information can be based on the doctor’s sole experience or based on existing health department summaries or any other piece of existing historical information.
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Overall rate of positives
The doctor can anticipate the overall rate of positive tests:

\[
P(X = 1) = P(X = 1 | \theta = 0)P(\theta = 0) + P(X = 1 | \theta = 1)P(\theta = 1)
\]

\[
= (0.4)(0.3) + (0.95)(0.7) = 0.785
\]
Turning the Bayesian crank

Once $X = 1$ is observed, i.e. once Diego is submitted to the test $X$ and the outcome is $X = 1$, what is the probability that Diego is ill?
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Common (and wrong!) answer: $P(X = 1|\theta = 1) = 0.95$

Correct answer: $P(\theta = 1|X = 1)$
Turning the Bayesian crank

Once $X = 1$ is observed, i.e. once Diego is submitted to the test $X$ and the outcome is $X = 1$, what is the probability that Diego is ill?

Common (and wrong!) answer: $P(X = 1|\theta = 1) = 0.95$

Correct answer: $P(\theta = 1|X = 1)$

Simple probability identity (Bayes’ rule):

$$P(\theta = 1|X = 1) = P(\theta = 1) \left\{ \frac{P(X = 1|\theta = 1)}{P(X = 1)} \right\}$$

$$= 0.70 \times \frac{0.95}{0.785}$$

$$= 0.70 \times 1.210191$$

$$= 0.847138$$
Combining both pieces of information

By combining

existing information (prior) + model/data (likelihood)

the updated (posterior) probability that Diego is ill is 85%.
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the updated (posterior) probability that Diego is ill is 85%.

More generally,

$$\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{predictive}}$$
What if instead $X = 0$?

**Maximum likelihood:**

$X = 0$ is more likely from a healthy patient than from an ill one

$$\frac{P(X = 0|\theta = 0)}{Pr(X = 0|\theta = 1)} = \frac{0.60}{0.05} = 12,$$

so MLE of $\theta$ is $\hat{\theta}_{MLE} = 0$.

**Bayes:**

Similarly, it is easy to see that

$$P(\theta = 0|X = 0) = P(\theta = 0) \left\{ \frac{P(X = 0|\theta = 0)}{P(X = 0)} \right\}$$

$$= 0.3 \times \frac{0.60}{0.215}$$

$$= 0.3 \times 2.790698$$

$$= 0.8373093$$
Sequential learning

The doctor is still not convinced and decides to perform a second more reliable test ($Y$):

$$P(Y = 0 | \theta = 1) = 0.01 \quad \text{versus} \quad P(X = 0 | \theta = 1) = 0.05$$

$$P(Y = 1 | \theta = 0) = 0.04 \quad \text{versus} \quad P(X = 1 | \theta = 0) = 0.40$$
Sequential learning

The doctor is still not convinced and decides to perform a second more reliable test \((Y)\):

\[
P(Y = 0|\theta = 1) = 0.01 \quad \text{versus} \quad P(X = 0|\theta = 1) = 0.05
\]
\[
P(Y = 1|\theta = 0) = 0.04 \quad \text{versus} \quad P(X = 1|\theta = 0) = 0.40
\]

Overall rate of positives

Once again, the doctor can anticipate the overall rate of positive tests, but now conditioning on \(X = 1\):

\[
P(Y = 1|X = 1) = P(Y = 1|\theta = 0)P(\theta = 0|X = 1) + P(Y = 1|\theta = 1)P(\theta = 1|X = 1)
\]
\[
= (0.04)(0.1528662) + (0.99)(0.8471338)
\]
\[
= 0.8447771
\]
Once again, Bayes rule leads to

\[
P(\theta = 1|X = 1, Y = 1) = P(\theta = 1|X = 1) \left\{ \frac{P(Y = 1|\theta = 1)}{P(Y = 1|X = 1)} \right\}
\]

\[
= 0.8471338 \times \frac{0.99}{0.8447771}
\]

\[
= 0.8471338 \times 1.171907
\]

\[
= 0.992762
\]
$Y = 1$ is observed

Once again, Bayes rule leads to

$$P(\theta = 1|X = 1, Y = 1) = P(\theta = 1|X = 1) \left\{ \frac{P(Y = 1|\theta = 1)}{P(Y = 1|X = 1)} \right\}$$

$$= 0.8471338 \times \frac{0.99}{0.8447771}$$

$$= 0.8471338 \times 1.171907$$

$$= 0.992762$$

Bayesian sequential learning:

$$P(\theta = 1|H) = \begin{cases} 
70\% & , H: \text{before } X \text{ and } Y \\
85\% & , H: \text{after } X = 1 \text{ and before } Y \\
99\% & , H: \text{after } X = 1 \text{ and } Y = 1
\end{cases}$$
$Y = 1$ is observed

Once again, Bayes rule leads to

$$P(\theta = 1 | X = 1, Y = 1) = P(\theta = 1 | X = 1) \left\{ \frac{P(Y = 1 | \theta = 1)}{P(Y = 1 | X = 1)} \right\}$$

$$= 0.8471338 \times \frac{0.99}{0.8447771}$$

$$= 0.8471338 \times 1.171907$$

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Bayesian sequential learning:

$$P(\theta = 1 | H) = \begin{cases} 70\% & , H: \text{before } X \text{ and } Y \\ 85\% & , H: \text{after } X = 1 \text{ and before } Y \\ 99\% & , H: \text{after } X = 1 \text{ and } Y = 1 \end{cases}$$

Note: It is easy to see that $Pr(\theta = 1 | Y = 1) = 98.2979\%$. 
$Y = 1$ is observed

Once again, Bayes rule leads to

$$P(\theta = 1 | X = 1, Y = 1) = P(\theta = 1 | X = 1) \left\{ \frac{P(Y = 1 | \theta = 1)}{P(Y = 1 | X = 1)} \right\}$$

$$= 0.8471338 \times \frac{0.99}{0.8447771}$$

$$= 0.8471338 \times 1.171907$$

$$= 0.992762$$

Bayesian sequential learning:

$$P(\theta = 1 | H) = \begin{cases} 70\% & , H: \text{before } X \text{ and } Y \\ 85\% & , H: \text{after } X = 1 \text{ and before } Y \\ 99\% & , H: \text{after } X = 1 \text{ and } Y = 1 \end{cases}$$

Note: It is easy to see that $Pr(\theta = 1 | Y = 1) = 98.2979\%$.

Conclusion: Don’t consider test $X$, unless it is “cost” free.
Example 2: Gaussian measurement error

**Goal:** Learn $\theta$, a physical quantity.
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Goal: Learn $\theta$, a physical quantity.
Measurement: $X$
Example 2: Gaussian measurement error

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Measurement: $X$
Model: $(X|\theta) \sim N(\theta, (40)^2)$
Example 2: Gaussian measurement error

Goal: Learn $\theta$, a physical quantity.
Measurement: $X$
Model: $(X|\theta) \sim N(\theta, (40)^2)$
$p(x | \theta)$ for $\theta \in \{600, 700, \ldots, 1000\}$
Large and small prior experience

Prior A: Physicist A (large experience): \( \theta \sim N(900, (20)^2) \)
Large and small prior experience

Prior A: Physicist A (large experience): $\theta \sim N(900, (20)^2)$
Prior B: Physicist B (not so experienced): $\theta \sim N(800, (80)^2)$
Large and small prior experience

Prior A: Physicist A (large experience): $\theta \sim N(900, (20)^2)$
Prior B: Physicist B (not so experienced): $\theta \sim N(800, (80)^2)$
Joint density: $p(x, \theta) = p(x|\theta)p(\theta)$
Bayesian computation: predictive

Prior: $\theta \sim N(\theta_0, \tau_0^2)$
Model: $x|\theta \sim N(\theta, \sigma^2)$

(Physicist A: $\theta_0 = 900$, $\tau_0 = 20$)

Predictive:

$$p(x) = \int_{-\infty}^{\infty} p(x|\theta)p(\theta)d\theta$$
Bayesian computation: predictive

Prior: \( \theta \sim N(\theta_0, \tau_0^2) \)  
(Physicist A: \( \theta_0 = 900, \tau_0 = 20 \))

Model: \( x|\theta \sim N(\theta, \sigma^2) \)

Predictive:

\[
p(x) = \int_{-\infty}^{\infty} p(x|\theta)p(\theta)\,d\theta
\]

Therefore,

\[
p(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\tau_0^2}} e^{-\frac{(\theta-\theta_0)^2}{2\tau_0^2}} \,d\theta
\]

\[
= \frac{1}{\sqrt{2\pi(\sigma^2 + \tau_0^2)}} e^{-\frac{(x-\theta_0)^2}{2(\sigma^2 + \tau_0^2)}}
\]

or

\[
x \sim N(\theta_0, \sigma^2 + \tau_0^2)
\]
Predictive densities
Bayesian computation: posterior

\[
p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)} \propto p(x|\theta)p(\theta)
\]

such that

\[
p(\theta|x) \propto (2\pi\sigma^2)^{-1/2} e^{-\frac{(x-\theta)^2}{2\sigma^2}} (2\pi\tau_0^2)^{-1/2} e^{-\frac{(\theta-\theta_0)^2}{2\tau_0^2}}
\]

\[
\propto \exp \left\{ -\frac{1}{2} \left[ \frac{(\theta^2 - 2\theta x)}{\sigma^2} + \frac{(\theta^2 - 2\theta \theta_0)}{\tau_0^2} \right] \right\}
\]

\[
\propto \exp \left\{ -\frac{1}{2\tau_1^2} (\theta - \theta_1)^2 \right\}.
\]

Therefore,

\[
\theta|x \sim N(\theta_1, \tau_1^2)
\]

where

\[
\theta_1 = \left( \frac{\sigma^2}{\sigma^2 + \tau_0^2} \right) \theta_0 + \left( \frac{\tau_0^2}{\sigma^2 + \tau_0^2} \right) x \quad \text{and} \quad \tau_1^2 = \tau_0^2 \left( \frac{\sigma^2}{\sigma^2 + \tau_0^2} \right)
\]
Combination of information

Let

\[ \pi = \frac{\sigma^2}{\sigma^2 + \tau^2_0} \in (0, 1) \]

Therefore,

\[ E(\theta|x) = \pi E(\theta) + (1 - \pi)x \]

and

\[ V(\theta|x) = \pi V(\theta) \]

When \( \tau^2_0 \) is much larger than \( \sigma^2 \), \( \pi \approx 0 \) and the posterior collapses at the observed value \( x \)!
Observation: $X = 850$
Posterior (updated) densities

Physicist A

Prior: $\theta \sim N(900, (20)^2)$
Posterior: $(\theta | X = 850) \sim N(890, (17.9)^2)$

Physicist B

Prior: $\theta \sim N(800, (40)^2)$
Posterior: $(\theta | X = 850) \sim N(840, (35.7)^2)$
Priors and posteriors

**Physicist A**

- Prior
- Posterior

**Physicist B**

- Prior
- Posterior
Summary

Deriving the posterior (via Bayes rule)

\[ p(\theta|x) \propto p(x|\theta)p(\theta) \]

and computing the predictive

\[ p(x) = \int_{\Theta} p(x|\theta)p(\theta)d\theta \]

can become very challenging!
Summary

Deriving the posterior (via Bayes rule)

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can become very challenging!

Bayesian computation was done on limited, unrealistic models until the Monte Carlo revolution (and the computing revolution) of the late 1980’s and early 1990’s.
A more conservative physicist

Prior A: Physicist A (large experience): \( \theta \sim N(900, 400) \)

Prior B: Physicist B (not so experienced): \( \theta \sim N(800, 1600) \)
A more conservative physicist

Prior A: Physicist A (large experience): $\theta \sim N(900, 400)$

Prior B: Physicist B (not so experienced): $\theta \sim N(800, 1600)$

Prior C: Physicist C (larger experience): $\theta \sim t_5(900, 240)$

$$V(Prior\ C) = \frac{5}{5 - 2}240 = 400 = V(Prior\ A)$$
Prior densities

![Graph showing prior densities for Physicist A, Physicist B, and Physicist C. The x-axis represents \( \theta \) and the y-axis represents density. The graph displays three peaks at different \( \theta \) values for each physicist.]
Closer look at the tails
Predictive and posterior of physicist C

For model \( x|\theta \sim N(\theta, \sigma^2) \) and prior of \( \theta \sim t_\nu(\theta_0, \tau^2) \),

\[
p(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right) \sqrt{\pi\nu\tau^2_0}} \left( 1 + \frac{1}{\nu} \left( \frac{\theta - \theta_0}{\tau_0} \right)^2 \right)^{-\frac{\nu+1}{2}} d\theta
\]

is not analytically available.

Similarly,

\[
p(\theta|x) \propto \exp \left\{ -\frac{(x - \theta)^2}{2\sigma^2} \right\} \left( 1 + \frac{1}{\nu} \left( \frac{\theta - \theta_0}{\tau_0} \right)^2 \right)^{-\frac{\nu+1}{2}}
\]

is of no known form.
Predictives

Monte Carlo approximation$^2$ to $p(x)$ for physicist C.

$^2$Yet to be learned!
Log predictives

Physicist C has similar knowledge as physicist A, but does not rule out smaller values for $x$. 

![Graph showing density vs. x for Physicist A, Physicist B, and Physicist C.](image_url)
Posteriors for θ

Monte Carlo approximation\(^3\) to \(p(\theta|x)\) for physicist C.

\(^3\)Yet to be learned!
Log posteriors

\[ \text{Log density} \]

\[ \theta \]

\[ \text{Physicist A} \]
\[ \text{Physicist C} \]
Monte Carlo: a toy example

In what follows, we will see how to approximate integrals and sample from unknown distributions via the well known *Monte Carlo* method.

Let us think about calculating $\pi = 3.141593 \ldots$

We could sample a bunch ($i = 1, \ldots, M$) of pairs $(x_i, y_i)$ in the unit square $(0,1) \times (0,1)$ and compute the fraction $\alpha$ of those pairs satisfying the condition $x_i^2 + y_i^2 < 1$. In this case, $\pi = 4\alpha$.

```r
M = 1000
x = runif(M)
y = runif(M)
cond = (x^2+y^2)<1
par(mfrow=c(1,2))
plot(x,y)
plot(x[cond],y[cond])
pi.mc = 4*sum(cond)/M
```
$$\pi_{mc} = 3.196$$

$$\frac{\pi}{4} = \int_0^1 \int_0^{\sqrt{1-x^2}} dydx$$
Monte Carlo: Let us play with $M$

```r
set.seed(12345)
M = 10000
x = runif(M)
y = runif(M)
cond = (x^2+y^2)<1
pi.mc = 4*cumsum(cond)/(1:M)
plot(1:M/1000,pi.mc,ylim=c(2.7,3.3),type="l",
     xlab="thousands of draws",ylab="pi approx.")
abline(h=pi,col=2)

for (i in 1:20){
  x = runif(M)
y = runif(M)
  cond = (x^2+y^2)<1
  pi.mc = 4*cumsum(cond)/(1:M)
  lines(1:M/1000,pi.mc,col=i)
}
```
MC error

The graph shows the approximation of π (pi) through Monte Carlo (MC) error over thousands of draws. The x-axis represents the number of thousands of draws, while the y-axis represents the approximated values of π. The values converge to around 3.1, illustrating the accuracy of the approximation method.