Model:

$X_1, X_2, \dots, X_n$ iid $Ber(\theta)$

- Prior 1: $\theta \sim \mathcal{U}(0,1)$ $\quad p(\theta) = 1 \quad \theta \in (0,1)$

- Prior 2: $\theta \sim Beta(a,b)$ $\quad p(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1} \quad \theta \in (0,1)$

- Prior 3: $\gamma = \log\left(\frac{\theta}{1-\theta}\right) \sim N(\mu, \sigma^2)$

(see next page) $p(\theta) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{\left(\log\left(\frac{\theta}{1-\theta}\right)-\mu\right)^2}{2\sigma^2}} \cdot \frac{1}{\theta(1-\theta)}$

$\theta \in (0,1)$

likelihood: $p(\underline{x}|\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$

$\sum_{i=1}^{n} x_i = 6, n = 10 \quad \Rightarrow \quad p(\underline{x}|\theta) = \theta^6(1-\theta)^4$

Prior 1: $p(\theta | \sum_{i=1}^{10} x_i = 6) \propto \theta^6(1-\theta)^4 \Rightarrow Beta(7,5)$

Prior 2: $p(\theta | \sum_{i=1}^{10} x_i = 6) \propto \theta^{(6+a)-1}(1-\theta)^{(4+b)-1} \Rightarrow Beta(6+a, 4+b)$

Prior 3: $p(\theta | \sum_{i=1}^{10} x_i = 6) \propto \theta^5(1-\theta)^3 \exp\left\{-\frac{1}{2\sigma^2}\left(\log\left(\frac{\theta}{1-\theta}\right)-\mu\right)^2\right\}$

Transformation:

$$\gamma = \log\left(\frac{\theta}{1-\theta}\right) \sim N(\mu, \sigma^2)$$

$\gamma = \log\theta - \log(1-\theta)$

$\frac{\partial\gamma}{\partial\theta} = \frac{1}{\theta} + \frac{1}{1-\theta}$

$$p_\theta(\theta) = p_\gamma\left(\log\frac{\theta}{1-\theta}\right)\left|\frac{\partial\gamma}{\partial\theta}\right|$$

$$= p_\gamma\left(\log\left(\frac{\theta}{1-\theta}\right)\right)\left|\frac{\partial\log\frac{\theta}{1-\theta}}{\partial\theta}\right|$$

$$= (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{\left(\log\frac{\theta}{1-\theta}-\mu\right)^2}{2\sigma^2}} \underbrace{\left|\frac{1}{\theta}+\frac{1}{1-\theta}\right|}_{\frac{\theta+(1-\theta)}{\theta(1-\theta)} = \frac{1}{\theta(1-\theta)}}$$

$$\boxed{p_\theta(\theta) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{\left(\log\left(\frac{\theta}{1-\theta}\right)-\mu\right)^2}{2\sigma^2}} \frac{1}{\theta(1-\theta)} \qquad \theta \in (0,1)}$$

$$E\left(\theta \,\middle|\, \sum_{i=1}^{10} x_i = 6, \text{prior } 1\right) = \frac{7}{7+5} = \frac{7}{12} = 0.58333$$

$$E\left(\theta \,\middle|\, \sum_{i=1}^{10} x_i = 6, \text{prior } 2\right) = \frac{6+a}{10+a+b} \overset{\substack{a=4 \\ b=2}}{=\!=} \frac{10}{16} \overset{0.625}{=} \cancel{0.6153846}$$

$$E\left(\theta \,\middle|\, \sum_{i=1}^{10} x_i = 6, \text{prior } 3\right) = \frac{\int_0^1 \theta^6 (1-\theta)^3 e^{-\frac{1}{2\sigma^2}\left(\log\left(\frac{\theta}{1-\theta}\right) - \mu\right)^2} d\theta}{\int_0^1 \theta^5 (1-\theta)^3 e^{-\frac{1}{2\sigma^2}\left(\log\left(\frac{\theta}{1-\theta}\right) - \mu\right)^2} d\theta}$$

$$\overset{\text{Approx.}}{=} \frac{0.001157192}{0.001912162} \cong 0.6051744$$

$\mu = 0$
$\sigma^2 = 3$