$$y_t = \alpha_{s_t} + \beta x_t + \epsilon_t \qquad \epsilon_t \sim N(0, \sigma^2) \qquad \theta = (\alpha_0, \alpha_1, \sigma^2, \pi, \xi)$$

$$\alpha_{s_t} = \begin{cases} \alpha_0 & s_t = 0 \\ \alpha_1 & s_t = 1 \end{cases} \qquad \begin{aligned} Pr(s_t = 1 \mid s_{t-1} = 1) &= \pi \\ Pr(s_t = 0 \mid s_{t-1} = 0) &= \xi \end{aligned}$$

$$D_t = \{y_{\tilde{t}}, x_{\tilde{t}}\}_{\tilde{t}=0}^{t}$$

$$S_0 \mid D_0 \sim Ber(\pi_0) \qquad (\pi_0 = 0.5, \text{ for instance})$$

$$Pr(S_1 = 1 \mid D_1) \propto p(y_1 \mid x_1, \theta, s_1 = 1) Pr(S_1 = 1 \mid D_0) \propto p_N(y_1; \alpha_1 + \beta x_1, \sigma^2) Pr(S_1 = 1 \mid D_0)$$

$$\begin{aligned} Pr(S_1 = 1 \mid D_0) &= Pr(S_1 = 1 \mid D_0, S_0 = 0) Pr(S_0 = 0 \mid D_0) + \\ &\quad Pr(S_1 = 1 \mid D_0, S_0 = 1) Pr(S_0 = 1 \mid D_0) \\ &= (1 - \xi)(1 - \pi_0) + \pi \pi_0 \end{aligned}$$

$$Pr(S_1 = 0 \mid D_1) \propto p_N(y_1; \alpha_0 + \beta x_1, \sigma^2) \underbrace{Pr(S_1 = 0 \mid D_0)}_{\xi(1-\pi_0) + (1-\pi)\pi_0}$$

$$\Rightarrow Pr(S_1 = 1 \mid D_1) = \pi_1 = \frac{p_N(y_1; \alpha_1 + \beta x_1, \sigma^2)\left[(1-\xi)(1-\pi_0) + \pi\pi_0\right]}{p_N(y_1; \alpha_1 + \beta x_1, \sigma^2)\left[(1-\xi)(1-\pi_0) + \pi\pi_0\right] + p_N(y_1; \alpha_0 + \beta x_1, \sigma^2)\left[\xi(1-\pi_0) + (1-\pi)\pi_0\right]}$$

Therefore, for $t = 2, 3, \ldots, n$

$$\Pr(S_{t-1}=1 \mid D_{t-1}) = \pi_{t-1} \qquad \left(S_{t-1} \mid D_{t-1} \sim \mathrm{Ber}(\pi_{t-1}) \right)$$

and

$$S_t \mid D_t \sim \mathrm{Ber}(\pi_t) \quad \text{where}$$

$$\pi_t = \frac{p_N(y_t; \alpha_1 + \beta x_t, \sigma^2)\left[(1-\xi)(1-\pi_{t-1}) + \pi\, \pi_{t-1}\right]}{p_N(y_t; \alpha_1 + \beta x_t, \sigma^2)\left[(1-\xi)(1-\pi_{t-1}) + \pi\, \pi_{t-1}\right] + p_N(y_t; \alpha_0 + \beta x_t, \sigma^2)\left[\xi(1-\pi_{t-1}) + \pi_{t-1}(1-\pi)\right]}$$

This is the forward filtering scheme. The Backward scheme works similarly:

$$\Pr(S_t = 1 \mid S_{t+1}=1, D_t) \propto \Pr(S_t = 1 \mid D_t)\Pr(S_{t+1}=1 \mid S_t = 1, D_t)$$
$$= \pi_t\, \pi$$
$$\Pr(S_t = 0 \mid S_{t+1}=1, D_t) \propto \Pr(S_t = 0 \mid D_t)\Pr(S_{t+1}=1 \mid S_t = 0)$$
$$= (1-\pi_t)(1-\xi)$$

$$\begin{cases} (S_t \mid S_{t+1}=1, D_t) \sim \mathrm{Ber}(\pi^B_{t,1}) & \pi^B_{t,1} = \dfrac{\pi_t\, \pi}{(1-\pi_t)(1-\xi) + \pi_t\, \pi} \\ (S_t \mid S_{t+1}=0, D_t) \sim \mathrm{Ber}(\pi^B_{t,0}) & \pi^B_{t,0} = \dfrac{\pi_t(1-\pi)}{\pi_t(1-\pi) + (1-\pi_t)\xi} \end{cases}$$