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PROBLEM 1

$$y_t = x_t + \epsilon_t \qquad \epsilon_t, \omega_t \text{ iid } N(0,1)$$
$$x_t = \phi x_{t-1} + \omega_t$$

(a) Assuming that $|\phi|<1$, it follows that the ACF of $\{x_t\}$ is given by $\rho_0^x = 1$, $\rho_1^x = \frac{\gamma_1}{\gamma_0}$, $\rho_2^x = \frac{\gamma_2}{\gamma_0}$, ..., where $\gamma_0^x = \frac{1}{1-\phi^2}$, $\gamma_\ell^x = \frac{\phi^\ell}{1-\phi^2}$ $\ell = \pm1, \pm2, \dots$. Therefore

$$\gamma_0^y = V(y_t) = V(x_t) + 1 = \gamma_0^x + 1 = \frac{2-\phi^2}{1-\phi^2}$$

$$\gamma_\ell^y = E(y_t y_{t-\ell}) = E((x_t+\epsilon_t)(x_{t-\ell} - \epsilon_{t-\ell})) = \gamma_\ell^x \quad \ell = \pm1, \pm2, \dots$$

$$\rightarrow \rho_\ell^x = \frac{\phi^\ell / 1-\phi^2}{(2-\phi^2)/(1-\phi^2)} = \frac{\phi^\ell}{2-\phi^\ell} \quad \ell = \pm1, \pm2, \dots$$

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(b)
$$E(y_{n+1} | y_{1:n}, \phi) = E(x_{n+1} + \epsilon_{n+1} | y_{1:n}, \phi)$$
$$= E(\phi x_n + \epsilon_{n+1} + \omega_{n+1} | y_{1:n}, \phi)$$
$$= \phi E(x_n | y_1, \dots, y_n, \phi) = \phi E(y_n - \epsilon_n | y_{1:n}, \phi)$$
$$= \phi y_n$$

$$E(y_{n+2} | y_{1:n}, \phi) = E[\phi x_{n+1} + \omega_{n+2} + \epsilon_{n+2} | y_{1:n}, \phi)$$
$$= \phi E(x_{n+1} | y_{1:n}, \phi) =$$
$$= \phi E(\phi x_n + \omega_{n+1} | y_{1:n}, \phi) =$$
$$= \phi^2 E(y_n - \epsilon_n | y_{1:n}, \phi) = \phi^2 y_n$$

(b) (continuing) ②

$$V(y_{n+1} | y_{1:n}, \phi) = V(\phi x_n + w_{n+1} + \epsilon_{n+1} | y_{1:n}, \phi)$$
$$= V(\phi(y_n - \epsilon_n) + w_{n+1} + \epsilon_{n+1} | y_{1:n}, \phi)$$
$$= \phi V(\epsilon_n) + V(w_{n+1}) + V(\epsilon_{n+1})$$
$$= \phi^2 + 2$$

$$V(y_{n+2} | y_{1:n}, \phi) = V(x_{n+2} + \epsilon_{n+2} | y_{1:n}, \phi)$$
$$= V(\phi x_{n+1} + w_{n+2} + \epsilon_{n+2} | y_{1:n}, \phi)$$
$$= V(\phi^2 x_n + \phi w_{n+1} + w_{n+2} + \epsilon_{n+2} | y_{1:n}, \phi)$$
$$= V(\phi^2 y_n - \phi^2 \epsilon_n + \phi w_{n+1} + w_{n+2} + \epsilon_{n+2} | y_{1:n}, \phi)$$
$$= \phi^4 + \phi^2 + 1 + 1$$
$$= \phi^4 + \phi^2 + 2$$

To sum up,

$\hat{y}_{n+1} = \phi y_n \quad e_{n+1} = y_{n+1} - \hat{y}_{n+1} \quad V(e_{n+1}) = 2 + \phi^2$

$\hat{y}_{n+2} = \phi^2 y_n \quad e_{n+2} = y_{n+2} - \hat{y}_{n+2} \quad V(e_{n+2}) = 2 + \phi^2 + \phi^4$

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(c) It is easy to see that

$$y_t = \epsilon_t + \sum_{j=0}^{t-1} \phi^j w_{t-j} \qquad \left(\text{recall that } x_0 = 0\right)$$

for $t = 1, 2, \ldots, n$. Therefore

$$E(y_t | \phi) = 0$$

$$V(y_t | \phi) = 1 + \sum_{j=0}^{t-1} \phi^{2j} = 1 + \frac{1-\phi^{2(t-1)}}{1-\phi^2}$$

How about $E(y_t y_{t+h} | \phi)$?

$$E(y_t y_{t+h}) = E\left\{\left(\epsilon_t + \sum_{j=0}^{t-1} \phi^j w_{t-j}\right)\left(\epsilon_{t+h} + \underbrace{\sum_{j=0}^{t+h-1} \phi^j w_{t+h-j}}\right)\right\}$$

$$= \sum_{j=0}^{h-1} \phi^j w_{t+h-j} + \sum_{j=h}^{t+h-1} \phi^j w_{t+h-j} = \sum_{j=0}^{h-1} \phi^j w_{t+h-j} + \sum_{j=0}^{t-1} \phi^{j+h} w_{t-j}$$

$$\Rightarrow E(y_t y_{t+h}) = E\left\{\left(\epsilon_t + \sum_{j=0}^{t-1} \phi^j w_{t-j}\right)\left(\epsilon_{t+h} + \sum_{j=0}^{h-1} \phi^j w_{t+h-j} + \sum_{j=0}^{t-1} \phi^{j+h} w_{t-j}\right)\right\}$$

$$= \sum_{j=0}^{t-1} \phi^{2j+h}$$

For instance,

$$E(y_1 y_2) = \phi \qquad E(y_2 y_3) = \phi + \phi^3$$
$$E(y_1 y_3) = \phi^2 \qquad E(y_2 y_4) = \phi^2 + \phi^4$$
$$\vdots \qquad\qquad \vdots$$
$$E(y_1 y_n) = \phi^{n-1} \qquad E(y_2 y_n) = \phi^{n-2} + \phi^n$$

Let $\Omega(\phi)$ be the $(n \times n)$ covariance matrix of $y_1, \cdots, y_n$, ie

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \Omega(\phi)\right].$$

Therefore, the likelihood function for ϕ is

$$L(\phi; y) = (2\pi)^{-\frac{n}{2}} |\Omega(\phi)|^{-1/2} \exp\left\{-\frac{1}{2} y^T \Omega^{-1}(\phi) y\right\}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Alternatively, one could notice that

$$x_1 = \phi x_0 + W_1$$
$$x_2 = \phi x_1 + W_2 \qquad\qquad x_0 = 0$$
$$\vdots$$
$$x_n = \phi x_{n-1} + W_n$$

could be written as

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\phi & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\phi & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & & -\phi & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_{n-1} \\ W_n \end{pmatrix}$$

$$A(\phi) X = W \qquad W \sim N(0, I_n)$$

$$\Rightarrow \quad X = A^{-1}(\phi) W \sim N\Big(0, \underbrace{A^{-1}(\phi)\left(A^{-1}(\phi)\right)^{T}}_{B(\phi)}\Big)$$

since $y = X + \epsilon$

$$\Rightarrow y \sim N\Big(0, \underbrace{I_n + B(\phi)}_{\Omega(\phi)}\Big)$$

PROBLEM 2

$$a_t = \begin{cases} u_t & t \text{ even} \\ \dfrac{u_{t-1}^2 - 1}{\sqrt{2}} & t \text{ odd} \end{cases}$$

$\{u_t\}$ iid $N(0,1)$

$u_t^2 \sim \chi_1^2 \quad E(u_t^2) = 1$

$V(u_t^2) = 2$

$$V(a_t) = \begin{cases} V(u_t) = 1 & t \text{ even} \\ V\left(\dfrac{u_{t-1}^2 - 1}{\sqrt{2}}\right) = \dfrac{1}{2} V(u_{t-1}^2) = \dfrac{2}{2} = 1 & t \text{ odd} \end{cases}$$

$$E(a_t) = \begin{cases} E(u_t) = 0 & t \text{ even} \\ \dfrac{E(u_{t-1}^2) - 1}{\sqrt{2}} = \dfrac{1-1}{\sqrt{2}} = 0 & t \text{ odd} \end{cases}$$

$$E(a_t a_{t+1}) = \begin{cases} E\left\{u_t \left(\dfrac{u_t^2 - 1}{\sqrt{2}}\right)\right\} = \dfrac{E(u_t^3) - E(u_t)}{\sqrt{2}} = \dfrac{0-0}{\sqrt{2}} = 0 & t \text{ even} \\ E\left(\left(\dfrac{u_{t-1}^2 - 1}{\sqrt{2}}\right) u_{t+1}\right) = 0 & t \text{ odd} \end{cases}$$

$$E(a_t a_{t+h}) = 0 \qquad h = \pm 2, \pm 3, \dots$$

Not only $\{a_t\}$ is weakly stationary, $\{a_t\}$ is also a white noise.