Unit root nonstationarity and long-memory

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Based on Tsay's *Analysis of Financial Time Series (3rd edition)*

Outline

Unit root nonstationarity

In some studies, interest rates, foreign exchange rates, or the price series of an asset are of interest.

These series tend to be nonstationary.

For a price series, the nonstationarity is mainly due to the fact that there is no fixed level for the price.

In the time series literature, such a nonstationary series is called unit-root nonstationary time series.

The best known example of unit-root nonstationary time series is the random-walk model.

Random walk

A time series $\{p_t\}$ is a random walk if it satisfies

$$p_t = p_{t-1} + a_t$$

where p_0 is a real number denoting the starting value of the process and $\{a_t\}$ is a white noise series.

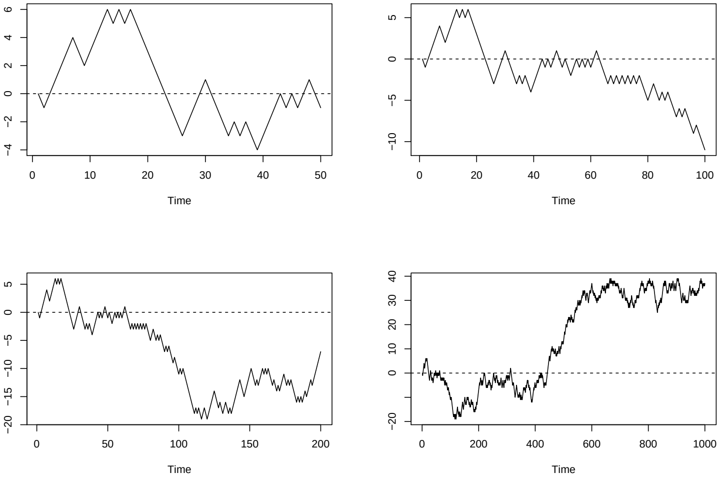
If p_t is the log price of a particular stock at date t , then p_0 could be the log price of the stock at its initial public offering (IPO) (i.e., the logged IPO price).

If at has a symmetric distribution around zero, then conditional on p_{t-1}, p_t has a 50-50 chance to go up or down, implying that pt would go up or down at random.

Example

```
set.seed(1234)
n=1000
p = rep(0,n)
a = rep(0,n)
p[1] = 0
a[1] = 1
for (t in 2:n){
  a[t] = sample(c(-1,1),size=1)
  p[t] = p[t-1] + a[t]
}

par(mfrow=c(2,2))
ts.plot(p[1:50],ylab="")
abline(h=0,lty=2)
ts.plot(p[1:100],ylab="")
abline(h=0,lty=2)
ts.plot(p[1:200],ylab="")
abline(h=0,lty=2)
ts.plot(p[1:1000],ylab="")
abline(h=0,lty=2)
```


6
4
2
0
−2
−4
0
10
20
30
40
50
Time
5
0
−5
−10
0
20
40
60
80
100
Time
5
0
−5
−10
−15
−20
0
50
100
150
200
Time
40
30
20
10
0
−20
0
200
400
600
800
1000
Time

AR(1) and random walk

If we treat the random-walk model as a special AR(1) model, then the coefficient of p_{t-1} is unity, which does not satisfy the weak stationarity condition of an AR(1) model.

A random-walk series is, therefore, not weakly stationary, and we call it a unit-root nonstationary time series.

The random-walk model has widely been considered as a statistical model for the movement of *logged stock prices.* Under such a model, the stock price is not predictable or mean reverting.

Mean reverting

The 1-step-ahead forecast of model $p_t = p_{t-1} + a_t$, at the forecast origin h is

$$\hat{p}_h(1) = E(p_{h+1}|p_h, p_{h-1}, \ldots) = p_h.$$

The 2-step-ahead forecast is

$$\hat{p}_h(2) = E(p_{h+2}|p_h, p_{h-1}, \ldots) = E(p_{h+1}|p_h, p_{h-1}, \ldots) = p_h,$$

which again is the log price at the forecast origin.

In fact, for any forecast horizon $l > 0$, we have

$$\hat{p}_h(l) = p_h$$

Therefore, the process is not mean reverting.

Predictability

The MA representation of the random-walk model is

$$p_t = a_t + a_{t-1} + a_{t-2} + \cdots$$

First, the l-step ahead forecast error is

$$\begin{aligned} e_h(l) &= p_{h+l} - \hat{p}_h(l) \\ &= p_{h+l} - p_h \\ &= a_{h+l} + a_{h+l-1} + \cdots + a_{h+1} \end{aligned}$$

so that

$$V[e_h(l)] = l\sigma_a^2 \to \infty \quad \text{as} \quad l \to \infty.$$

This result says that the usefulness of point forecast $\hat{p}_h(l)$ diminishes as l increases, which again implies that the model is not predictable.

Memory

The impact of any past shock a_{t-i} on pt does not decay over time.

The series has a strong memory as it remembers all of the past shocks.

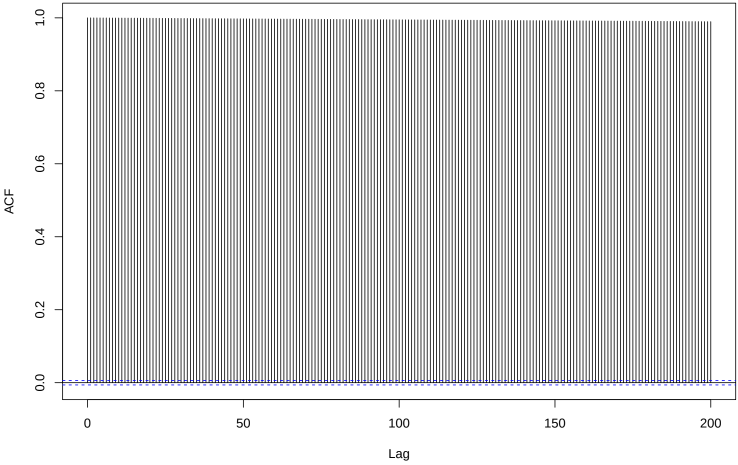
In economics, the shocks are said to have a permanent effect on the series.

The strong memory of a unit-root time series can be seen from the sample ACF of the observed series.

The sample ACFs are all approaching 1 as the sample size increases.

ACF

Based on 100,000 observations!



Random walk with drift

Log return series of a market index tends to have a small and positive mean, so the model for the log price is

$$p_t = \mu + p_{t-1} + a_t$$

The constant term μ represents the time trend of the log price p_t and is often referred to as the *drift of the model.*

Assume that the initial log price is p_0:

$$\begin{aligned}
p_1 &= \mu + p_0 + a_1, \\
p_2 &= \mu + p_1 + a_2 = 2\mu + p_0 + a_2 + a_1, \\
p_3 &= \mu + p_2 + a_3 = 3\mu + p_0 + a_3 + a_2 + a_1, \\
&\vdots \\
p_t &= \mu t + p_0 + \sum_{i=1}^{t} a_i.
\end{aligned}$$

The log price consists of a time trend and a pure random-walk process:

$$p_t = p_0 + \mu t + \sum_{i=1}^{t} a_i.$$

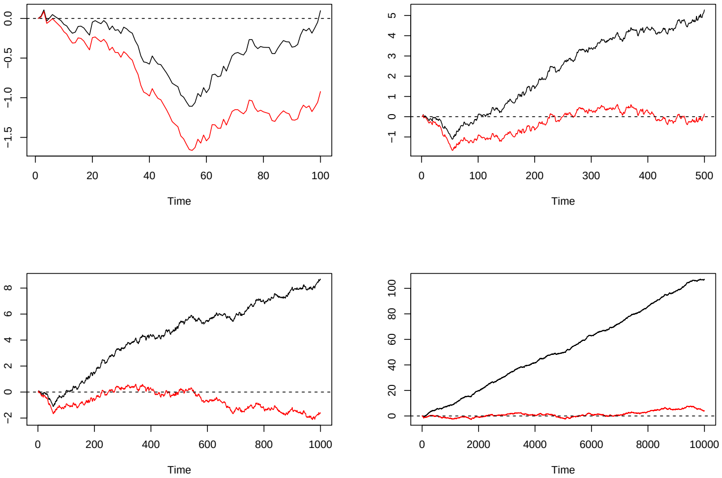
The conditional standard deviation of p_t, $\sigma_a \sqrt{t}$, grows at a slower rate than the conditional expectation of p_t, $p_0 + \mu t$.

Let $n = 10,000$ and a_t iid $N(0, 0.0637^2)$ so

$$p_t = \mu + p_{t-1} + a_t$$

where $p_0 = 0$ and $\mu = 0$ or $\mu = 0.0103$.

RW with drift



Interpretation of the Constant Term

MA(q) model: the constant term is the mean of the series.

Stationary AR(p) or ARMA(p,q) models: the constant term is related to the mean via

$$\mu = \frac{\phi_0}{1 - \phi_1 - \cdots - \phi_p}.$$

Random walk with drift: the constant term becomes the time slope of the series.

Trend-stationarity

The simplest trend-stationary time series model,

$$p_t = \beta_0 + \beta_1 t + r_t,$$

where r_t is a stationary time series, say a stationary AR(p).

Major difference between the two models:

- Random walk with drift model

$$E(p_t) = p_0 + \mu t \quad \text{and} \quad V(p_t) = \sigma_a^2 t.$$

- Trend-stationary model

$$E(p_t) = \beta_0 + \beta_1 t \quad \text{and} \quad V(p_t) = V(r_t)$$

with $V(r_t)$ finite and time invariant.

General unit-root models

Consider an ARMA model. If one extends the model by allowing the AR polynomial to have 1 as a characteristic root, then the model becomes the well-known autoregressive integrated moving-average (ARIMA) model.

An ARIMA model is said to be unit-root nonstationary because its AR polynomial has a unit root.

Like a random-walk model, an ARIMA model has strong memory because the ψ_i coefficients in its MA representation do not decay over time to zero, implying that the past shock a_{t-i} of the model has a *permanent effect* on the series.

ARIMA$(p, 1, q)$

A conventional approach for handling unit-root nonstationarity is to use *differencing*.

A time series y_t is said to be an ARIMA$(p, 1, q)$ process if the change series

$$c_t = y_t - y_{t-1} = (1 - B)y_t$$

follows a stationary and invertible ARMA(p, q) model.

Price series are believed to be nonstationary, but the log return series,

$$r_t = \log(P_t) - \log(P_{t-1}),$$

is stationary.

Unit-root test

To test whether the log price p_t of an asset follows a random walk or a random walk with drift, we employ the models

$$p_t = \phi_1 p_{t-1} + e_t \quad (1)$$

$$p_t = \phi_0 + \phi_1 p_{t-1} + e_t \quad (2)$$

where e_t denotes the error term, and consider the null hypothesis

$$H_0 : \phi_1 = 1$$

versus the alternative hypothesis

$$H_a : \phi_1 < 1.$$

This is the well-known *unit-root testing problem* (Dickey and Fuller, 1979).

Dickey-Fuller test

A convenient test statistic is the t ratio of the least-squares (LS) estimate of ϕ_1 under the null hypothesis.

For equation (1), the LS method gives

$$\hat{\phi}_1 = \frac{\sum_{t=1}^{T} p_{t-1} p_t}{\sum_{t=1}^{T} p_{t-1}^2} \quad \text{and} \quad \hat{\sigma}_e^2 = \frac{\sum_{t=1}^{T} (p_t - \hat{\phi}_1 p_{t-1})^2}{T-1}$$

where $p_0 = 0$. The t ratio is

$$\text{DF} \equiv t \text{ ratio} = \frac{\hat{\phi}_1 - 1}{\text{std}(\hat{\phi}_1)} = \frac{\sum_{t=1}^{T} p_{t-1} e_t}{\hat{\sigma}_e \sqrt{\sum_{t=1}^{T} p_{t-1}^2}}$$

which is commonly referred to as the Dickey-Fuller (DF) test.

If $\{e_t\}$ is a white noise series with finite moments of order slightly greater than 2, then the DF statistic converges to a function of the standard Brownian motion as $T \to \infty$; see Chan and Wei (1988) and Phillips (1987) for more information.

If $\phi_0 = 0$ but equation (2) is employed anyway, then the resulting t ratio for testing $\phi_1 = 1$ will converge to another nonstandard asymptotic distribution.

If $\phi_0 \neq 0$ and equation (2) is used, then the t ratio for testing $\phi_1 = 1$ is asymptotically normal.

An AR(p) model

$$y_t = \phi_0 + \sum_{i=1}^{p} \phi_i y_{t-i} + e_t,$$

or

$$(1 - \phi_1 B - \phi_2 B^2 + \cdots \phi_p B^p) y_t = \Phi(B) y_t = \phi_0 + e_t,$$

can be rewritten as

$$\Delta y_t = \pi x_{t-1} + \sum_{i=1}^{p-1} \tilde{\phi}_i \Delta y_{t-i} + e_t,$$

where

- $\Delta y_t = (1 - B) y_t = y_t - y_{t-1}$,
- $\pi = -\Phi(1) = \phi_1 + \phi_2 + \cdots + \phi_p - 1$,
- $\tilde{\phi}$s are functions of the ϕs.

A (single) unit root implies that $\Phi(1) = \pi = 0$ or that

$$\phi_1 + \phi_2 + \cdots + \phi_p = 1.$$

AR(3) model

The model

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + e_t,$$

can be rewriten as

$$\Delta y_t = \pi x_{t-1} + \tilde{\phi}_2 \Delta y_{t-2} + \tilde{\phi}_2 \Delta y_{t-2} + e_t,$$

where

$$\begin{aligned} \pi &= \phi_1 + \phi_2 + \phi_3 - 1 \\ \tilde{\phi}_1 &= -(\phi_2 + \phi_3) \\ \tilde{\phi}_2 &= -\phi_3 \end{aligned}$$

Augmented DF test

To verify the existence of a unit root in an AR(p) process, one may perform the test

$$H_0 : \beta_1 = 1 \quad \text{versus} \quad H_a : \beta < 1$$

using the regression

$$y_t = c_t + \beta y_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta y_{t-i} + e_t,$$

$\Delta y_t = y_t - y_{t-1}$ and

$$c_t = \begin{cases} 0 & \text{no constant, no trend} \\ \omega_0 \neq 0 & \text{constant, no trend} \\ \omega_0 + \omega_1 t & \text{constant, trend} \end{cases}$$

The t ratio of $\hat{\beta} - 1$,

$$\text{ADF-test} = \frac{\hat{\beta} - 1}{\text{std}(\hat{\beta})}$$

where $\hat{\beta}$ denotes the least-squares estimate of β, is the well-known augmented Dickey-Fuller (ADF) unit-root test.

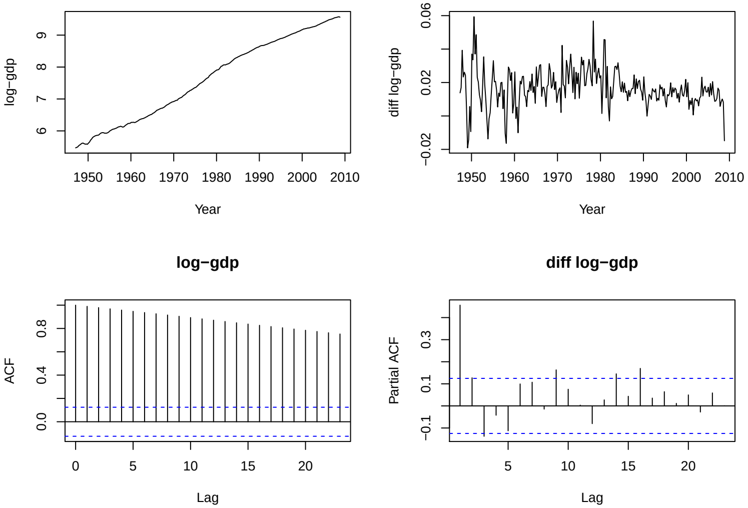
US quarterly GDP

Consider the log series of U.S. quarterly GDP from 1947.I to 2008.IV.

The series exhibits an upward trend, showing the growth of the U.S. economy, and has high sample serial correlations.

The first differenced series, representing the growth rate of U.S. GDP, seems to vary around a fixed mean level, even though the variability appears to be smaller in recent years.

With $p = 10$, the ADF test statistic is -1.61 with a p-value 0.45, indicating that the unit-root hypothesis cannot be rejected.


log−gdp
Year
diff log−gdp
Year
log−gdp
ACF
Lag
diff log−gdp
Partial ACF
Lag

R code

```
install.packages("tseries")
library("tseries")
da=read.table("http://faculty.chicagobooth.edu/ruey.tsay/teaching/fts3/q-gdp4708.txt",header=T)
n  = nrow(da)
da[c(1,n),]
#  year mon day   gdp
#1   1947   1   1   237.2
#248 2008  10   1 14200.3

gdp=log(da[,4])
date = da[,1]+da[,2]/12

par(mfrow=c(2,2))
plot(date,gdp,xlab="Year",ylab="log-gdp",type="l")
plot(date[2:n],diff(gdp),xlab="Year",ylab="diff log-gdp",type="l")
acf(gdp,main="log-gdp")
pacf(diff(gdp),main="diff log-gdp")

m1=ar(diff(gdp),method="mle")
m1
#Coefficients:
#      1        2        3        4        5        6        7        8        9       10
# 0.4045   0.1883  -0.1158   0.0215  -0.1602   0.0739   0.0853  -0.0993   0.1538   0.0959
#
#Order selected 10  sigma^2 estimated as  8.68e-05

adf.test(gdp,k=10)
#
# Augmented Dickey-Fuller Test
#data:  gdp
#Dickey-Fuller = -0.37047, Lag order = 10, p-value = 0.9872
#alternative hypothesis: stationary
```

S&P500 returns

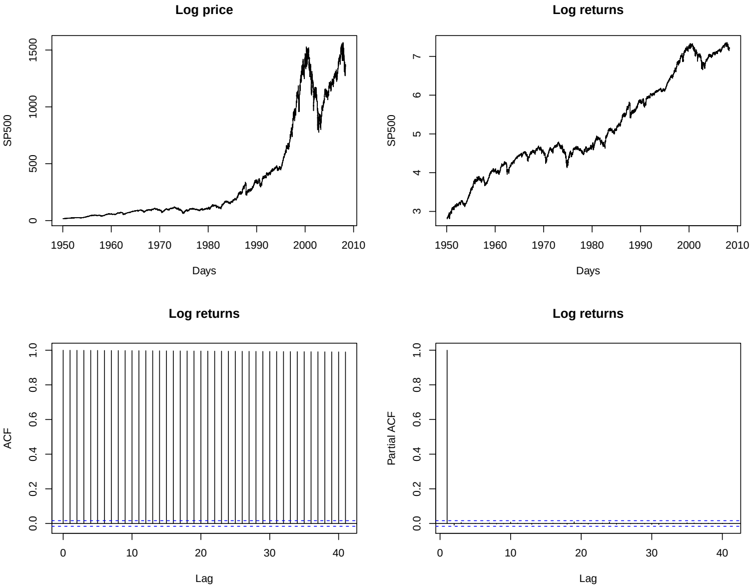
Consider the log series of the S&P 500 index from January 3, 1950, to April 16, 2008, for 14,462 observations.

Testing for a unit root in the index is relevant if one wishes to verify empirically that the index follows a random walk with drift. To this end, we use $c_t = \omega_0 + \omega_1 t$ in applying the ADF test.

Furthermore, we choose $p = 15$ based on the sample PACF of the first differenced series.

The resulting test statistic is -1.998 with a p-value of 0.602.

Thus, the unit-root hypothesis cannot be rejected at any reasonable significance level.


Log price
SP500
Days
Log returns
SP500
Days
Log returns
ACF
Lag
Log returns
Partial ACF
Lag

```
install.packages("tseries")
library("tseries")
da=read.table("http://faculty.chicagobooth.edu/ruey.tsay/teaching/fts3/d-sp55008.txt",header=T)
sp5=log(da[,7])
date = da[,1]+(da[,2]*30+da[,3]/31)/366

pdf(file="adftest1.pdf",width=10,height=8)
par(mfrow=c(2,2))
plot(date,da[,7],xlab="Days",ylab="SP500",type="l",main="Log price")
plot(date,sp5,xlab="Days",ylab="SP500",type="l",main="Log returns")
acf(sp5,main="Log returns")
pacf(sp5,main="Log returns")
dev.off()

adf.test(sp5,k=10)
# Augmented Dickey-Fuller Test
#data:  sp5
#Dickey-Fuller = -1.9482, Lag order = 10, p-value = 0.6004
#alternative hypothesis: stationary

m2=ar(diff(sp5),method="mle")
m2$x.mean
#[1] 0.0002988915
m2
#Call:
#ar(x = diff(sp5), method = "mle")
#
#Coefficients:
#     1        2
#0.0721  -0.0387
#
#Order selected 2  sigma^2 estimated as  8.068e-05

adf.test(diff(sp5),k=10)
# Augmented Dickey-Fuller Test
#data:  diff(sp5)
#Dickey-Fuller = -37.537, Lag order = 10, p-value = 0.01
#alternative hypothesis: stationary
```

Long memory models

There exist some time series whose ACF decays slowly to zero at a polynomial rate as the lag increases. These processes are referred to as *long-memory time series*.

One such example is the fractionally differenced process defined by

$$(1-B)^d x_t = a_t \qquad -0.5 < d < 0.5,$$

where $\{a_t\}$ is a white noise series, and

$$(1-B)^d = \sum_{k=0}^{\infty}(-1)^k \frac{d(d-1)\cdots(d-k+1)}{k!} B^k$$

History

The study of long memory originated in the 1950s in the field of hydrology, where studies of the levels of the river Nile (Hurst, 1951) demonstrated anomalously fast growth of the rescaled range of the time series.

After protracted debates1 about whether this was a transient (finite time) effect, the mathematical pioneer Benoît B. Mandelbrot showed that if one retained the assumption of stationarity, novel mathematics would then be essential to sufficiently explain the Hurst effect.

In doing so he rigorously defined (Mandelbrot and Van Ness, 1968; Mandelbrot and Wallis, 1968) the concept of long memory.

Palma (2007) *Long Memory Time Series*. Wiley.
Beran, Feng, Ghosh, Kulik (2013) *Long Memory Processes*. Springer.

Some properties

1. If $d < 0.5$, then x_t is a weakly stationary process and has the infinite MA representation

$$x_t = a_t + \sum_{i=1}^{\infty} \psi_i a_{t-i}$$

where

$$\psi_k = \frac{(k+d-1)!}{k!(d-1)!}$$

2. If $d > -0.5$, then x_t is invertible and has the infinite AR representation

$$x_t = \sum_{i=1}^{\infty} \pi_i x_{t-i} + a_t$$

where

$$\pi_k = \frac{(k-d-1)!}{k!(-d-1)!}$$

3. For $-0.5 < d < 0.5$, the ACF of x_t is

$$\rho_k = \frac{d(1+d)\cdots(k-1+d)}{(1-d)(2-d)\cdots(k=d)}, \qquad k = 1, 2, \ldots.$$

In particular, $\rho_1 = d/(1-d)$ and

$$\rho_k \approx \frac{(-d)!}{(d-1)!} k^{2d-1} \quad \text{as} \quad k \to \infty.$$

4. For $-0.5 < d < 0.5$, the PACF of x_t is $\phi_{k,k} = d/(k-d)$ for $k = 1, 2, \ldots$.

5. For $-0.5 < d < 0.5$, the spectral density function $f(\omega)$ of x_t, which is the Fourier transform of the ACF of x_t, satisfies

$$f(\omega) \sim \omega^{-2d}, \quad \text{as } \omega \to 0,$$

where $\omega \in [0, 2\pi]$ denotes the frequency.

Of particular interest here is the behavior of ACF of x_t when $d < 0.5$.

The property says that $\rho_k \sim k^{2d-1}$, which decays at a polynomial, instead of exponential, rate.

For this reason, such an x_t process is called a long-memory time series.

A special characteristic of the spectral density function is that the spectrum diverges to infinity as $\omega \to 0$.

However, the spectral density function of a stationary ARMA process is bounded for all $\omega \in [0, 2\pi]$.

If the fractionally differenced series $(1 - B)^d x_t$ follows an ARMA(p, q) model, then x_t is called an ARFIMA(p, d, q) process, which is a generalized ARIMA model by allowing for noninteger d.

In practice, if the sample ACF of a time series is not large in magnitude, but decays slowly, then the series may have long memory.

For the pure fractionally differenced model, one can estimate d using either a maximum-likelihood method or a regression method with logged periodogram at the lower frequencies.

Bayesian ARFIMA

Koop, Ley, Osiewalski and Steel (1997) Bayesian analysis of long memory and persistence using ARFIMA models. *Journal of Econometrics*, **76**, 149-169.

Pai and Ravishanker (1998) Bayesian analysis of autoregressive fractionally integrated moving-average processes. *Journal of Time Series Analysis*, **19**, 99-112.

Liseo, Marinucci and Petrella (2001) Bayesian semiparametric inference on long-range dependence. *Biometrika*, **88**, 1089-1104.

Ko and Vannucci (2006) Bayesian wavelet analysis of ARFIMA processes. *Journal of Statistical Planning and Inference*, **136**, 3415-3434.

Ko and Vannucci (2006) Bayesian Wavelet-Based Methods for the Detection of Multiple Changes of the Long Memory Parameter. *IEEE Transations on Signal Processing*, **54**, 11, 4461.

Holan, McElroy and Chakraborty (2009) A Bayesian Approach to Estimating the Long Memory Parameter. *Bayesian Analysis*, **4**, 159-190.

Graves (2013) *A systematic approach to Bayesian inference for long memory processes.* Ph.D. thesis, University of Cambridge, UK.

Graves, Gramacy, Watkins and Franzke (2014) A brief history of long memory: Hurst, Mandelbrot and the road to ARFIMA. `http://arxiv.org/abs/1406.6018`

Graves, Gramacy, Franzke and Watkins (2015) Efficient Bayesian inference for ARFIMA processes. *Nonlinear Processes in Geophysics*, **22**, 679-700.
`http://www.nonlin-processes-geophys.net/22/679/2015/npg-22-679-2015.pdf`