Put Option Implied Risk-Premia in General Equilibrium under Recursive Preferences

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Abstract

We show that the large time-series variation of put option prices on the S&P 500 Futures Index can be understood through traditional asset-pricing theory. This phenomenon is explained through a general equilibrium model based on recursive preferences, linear production technology and time-varying production growth rates. We directly take our general equilibrium framework to structural estimation using Bayesian methodologies and filter the underlying economic states and estimate the structural parameters from a panel of put option prices and consumption growth dynamics. The model implied risk-premium estimated from put options is large, counter-cyclical and captured through the long-run risk channel of recursive preferences, however, with short-run dynamics. We distinguish ourselves from other long-run risk models that rely on exogenous specifications of stochastic volatility to generate time-variation in risk-premium. We rely on the non-linearity of the endogenously determined consumption-to-capital ratio that generates strong precautionary savings in poor economic states. This provides us an useful channel to study insurance motivation and explain why put option prices are highly counter-cyclical. The Bayesian estimation technique gives us the computational flexibility to estimate structural parameters and filter a latent state variable from a highly non-linear state-space.

Keywords: Recursive Preferences, Risk-Premia, Put Options, Bayesian MCMC, Non-linear Filtering.

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1 Introduction

Put option prices reveal interesting dynamics of underlying risk preferences in the economy. The time-series evidence suggests that put option prices are highly counter-cyclical. Prices rise in response to signals of poor economic growth. Similarly, prices fall in good times when growth rate of the economy is strong. Figure 1 shows high put option prices[1] in response to negative shocks to economic growth (e.g. Russian Ruble crisis, September 11th, NBER peak, etc.), whereas the economic expansion in the mid-1990's was marked with low put option prices. It shows willingness to pay more to protect wealth invested in the economy when underlying economic states are poor, whereas prices of these contracts fall in good times.

Bloom (2009) also analyzes the above phenomenon[2] and provides a link between such spikes in the put option prices and subsequent decline in economic activity. Motivated by Bloom (2009), we seek to explain the above time-series variation in put option prices by providing a link between the state of economic activity and the risk-premium in those states. In doing so, we explain the seemingly puzzling fact that put options are overpriced relative to canonical asset-pricing models, like CAPM or Rubinstein (1976). For example, Bondarenko (2003) points out between 1987 and 2000 the average excess return for one-month maturity put options range from -39% for at-the-money and -95% for deep out-of-the-money options. This evidence suggests that an agent can make large profits from selling unhedged put options. Our model shows we can generate high risk-premia in poor economic states out of strong preference for precautionary savings that drives insurance motivation and generates high put option prices that we observe in the data. Similarly, in good economic states precautionary savings dissipates, and the risk-premia falls resulting in low put option prices.

Our solution is in a general equilibrium framework under stochastic differential utility (SDU) of Duffie and Epstein (1992) in a single-good economy characterized by a linear production function and time-varying productivity growth. In our stylized set-up, the volatility of the "underlying" wealth process on which put options are written is constant. However, we explain the time-series pattern of put option prices with counter-cyclical risk-premia, which is a compensation for the shock to marginal utility from exposure to time-varying production fluctations. We show that the put-option implied risk-premia under SDU is as high as 30% in bad states to -10% under good states. Additionally, since we are

[1]The figure shows the time-series of relative put option prices, where relative price is defined by the price of the put option relative to the value of the underlying S&P 500 Futures Index.

[2]The evidence that Bloom (2009) cites is with respect to the VIX index which has a one-to-one relationship with relative put option prices.

able to fit a full panel of put option prices, we are also able to match the put option smirk generated by the data. The time-series variation in prices of these options is brought about by the time-variation in compensation of risk due to exposure to stochastic growth rates and SDU.

This paper makes two important contributions. First, it generates counter-cyclical risk-premia from time-varying productivity growth rates. In the model, the underlying dynamics and preferences interact to produce time-variation in shocks to marginal utility which gives rise to time-varying risk-premia. A long line of asset-pricing literature under recursive preferences, starting with Bansal and Yaron (2004), relies on stochastic volatility in the dynamics of the economic fundamentals to generate stochastic risk-premia. We can generate counter-cyclical premia under much simpler dynamics of the economic fundamentals. In addition, we directly take our general equilibrium theory to Bayesian estimation and filter the underlying growth rates from a full panel of option prices using a flexible non-linear filtering methodology. We also estimate structural parameters of the underlying preferences and dynamics from put option prices and consumption growth.

To explain the counter-cyclical risk-premia that generates the time-variation in put option prices, we start from the underlying source of risk in this economy. The origin of time-varying risk-premia is the endogenously determined log consumption-to-capital ratio that is non-linear in the underlying growth rate. The non-linearity arises from the fact that our solution of the value function is quadratic in the growth rate. This non-linearity has important economic consequences. For example, it implies that the magnitude of the income or substitution effects in this economy will be determined by the underlying economic state. This time variation in income and substitution effects imply that the agent will interpret a negative shock in the growth rate very differently depending on whether the prevailing economic state is high or low. When the growth rate of the economy is lower than the average growth rate and the economy receives a negative shock, we see the income effect dominating the substitution effect. We generate high risk-premia in these states that generates high no-arbitrage put option prices. Similarly, when growth rates are above average, the substitution effect starts to dominate the income effect, even with intertemporal elasticity of substitution (IES) less than one. In these states, the agent responds to a negative shock in the growth rate by increasing consumption. Clearly the agent's precautionary savings motivation has dissipated, and the risk-premia is low in these states. Naturally, the no-arbitrage put option prices are also low. The utilization of SDU is crucial in our setting. First, we can generate extremely high risk-premia to match put option prices from reasonably low risk-aversion. Secondly, the time-variation in risk-premia is solely due to IES less than one, and in fact, the risk-premia becomes constant for IES greater than or equal to one. Finally, we can also generate consumption growth dynamics that are close to what we observe in the data due to the separation of risk-preferences from intertemporal substitution that SDU offers.

It is crucial to point out that the fluctuation between income and substitution effects across low or high states produces a significant variation in the price of certainty equivalence. In poor states, the price of certainty equivalence is high (i.e., the risk-free rate is low), but the income effect guarantees that the savings motivation of the agent still prevails with a negative growth rate shock. Thus, the income effect is really a strong preference for precautionary savings. Precisely the opposite takes place in good economic states when precautionary savings motive disappears, the price of certainty equivalence falls and the substitution effect ensures that consumption rises relative to savings. It is noteworthy to mention that this time-variation in the price of certainty equivalence creates an unfortunately large volatility of the risk-free rate.

We address the econometric challenge posed in this problem through a flexible Bayesian estimation methodology. First of all, we filter the underlying economic growth rates from a panel of put option prices - a highly non-linear filtering exercise, using Markov Chain Monte Carlo on a non-linear state-space model. Estimation of structural parameters from a panel of put options is also challenging because they enter non-linearly through the underlying risk-neutral dynamics into put option prices. Thus, in order to estimate these structural parameters - risk aversion, autocorrelation of growth rates, long-term mean of log consumption to capital ratio, etc. we follow a Metropolis-Hastings procedure. Since our claim is that we are doing general equilibrium analysis, we also analyze aggregate consumption growth implied by our model. We use the posterior distribution of parameters obtained from put option prices as strong prior information on those same parameters when we consider consumption growth. We estimate the IES and time-preference parameters from consumption growth data as these parameters do not enter option prices, and generate reasonable filtered and smoothed estimates of expected consumption growth (Figure 10). Alas, the non-linearity in the log consumption-to-capital ratio under recursive preferences implies very high volatility in the price of certainty equivalence. The early resolution of uncertainty that SDU offers under certain parametric restriction also brings forth large certainty equivalent fluctuations resulting in high volatility of the risk-free rate.

This paper adds to the growing body of literature on equilibrium option pricing by proposing a real business cycle paradigm with recursive preferences and counter-cyclical premia. The papers closest to this paper are Benzoni, Collin-Dufresne and Goldstein (2005), Eraker and Shaliastovich (2008), Drechsler and Yaron (2007) and Shaliastovich (2008). We separate ourselves from the current literature in many ways - first ours is a complete general equilibrium set-up where consumption is endogenous. Second, we address the time-series pattern of put option prices from endogenously determined counter-cyclical premia and finally our dynamics is entirely Gaussian. Moreover, we subject ourselves to a formal empirical scrutiny and estimate our model from a panel of put options to produce the underlying structural parameters and growth rates.

The papers listed above, except for Shaliastovich (2008), all rely on hard-wired jump components in underlying partial-equilibrium dynamics to generate high risk-premia. Eraker and Shaliastovich (2008) shows large jumps in consumption volatility translates into significant premium for OTM put options. Benzoni, et.al. (2005) and Drechsler and Yaron (2007) both model jumps in expected consumption growth rate (and also conditional volatility in the latter case). In both these models the jump dynamics are absolutely essential in matching the observed pattern in put option prices. Benzoni, et.al. (2005) show that they can match the level of the put option smile with jump dynamics under recursive preferences, and Drechsler and Yaron (2007) show that their jump dynamics in conditional moments of consumption growth can capture the variance premium, which is the difference between option-implied risk-neutral variance and observed return variance. They further show that without the jump components their model cannot come close to matching the variance premium. Shaliastovich (2008) has two different channels - one is risk from short-run dynamics (whose effect is small) and a learning dynamics. Together, they can generate a steep smile for OTM put options.

The path taken by us is very different. Our analysis is entirely general equilibrium in a single-good economy. Our model is similar to Ai (2009) where the agent learns about the underlying state. We look at a full information version of Ai (2009) and our solution is also quite different because we focus on the non-linearity of the value function. The nature of our solution implies that the time-variation in risk-premia is brought about by short-run, i.e. not strongly autocorrelated, dynamics of the underlying growth rates of the economy. Naturally, the risk-premia varies significantly if the underlying states are less autocorrelated. Thus, we can essentially bring about the same features of "jumps" using short-run dynamics. The fact that strongly autocorrelated dynamics can generate substantial premia goes back to Bansal and Yaron (2004). A large autocorrelation parameter generates high volatility, and eventually high premia, out of growth rates. Indeed, in Drechsler and Yaron (2007) and Shaliastovich (2008), a significant portion of the return premia is driven by strongly autocorrelated states. Our findings are very different. In fact, our solution of the value function implies that we *need* the underlying states to be far less autocorrelated to generate economically significant risk-premia. Thus, the long-run risk due to SDU is really from short-run dynamics! Furthermore, we infer risk-aversion from put option prices to be between 2.1-9.8 (with a median of 4.8) and a long run average of consumption to capital ratio between 0.01-0.13 (with a median of 0.10) which contains the consumption to wealth ratio computed in Lustig, et. al. (2007). Furthermore, the median IES estimate is less than one which is consistent with IES estimates from Hall (1988), Campbell (1999) and Vissing-Jorgensen (2002).

Other equilibrium papers on option pricing such as Liu, Pan and Wang (2005) offers an explanation of OTM put options by considering model uncertainty a la Anderson, Hansen, and Sargent (2003). In their model, the agent exhibits uncertainty about the distribution of jumps in return and can generate steeper volatility smirks due to this uncertainty. Brown and Jackwerth (2004) consider a setting where the pricing kernel of the agent has a stochastic component driven by a "momentum" state variable. Bates (2001) proposes a model where agents exhibit crash aversion to capture many observed facts from the options market. Constantinides, Jackwerth and Perrakis (2006) documents violation of stochastic dominance in out-of-the-money calls under incomplete market and generous transaction costs.

Our paper also fits into the body of literature that proposes alternative to the iid assumption of Black-Scholes by extending the underlying returns process to stochastic volatility (Heston (1993)) and stochastic volatility and jumps (Bates (1996)). Duffie, Pan and Singleton (2000) show that under affine dynamics a host of valuation problems can be analyzed through a class of transforms. Our explanation is based on a traditional real-business cycle setting that generates counter-cyclical premia under stochastic production growth rates and SDU. In our case, the option valuation problem is strictly affine and thus the standard solutions of Duffie, Pan and Singleton (2000) apply directly and we can express the option pricing problem as an inverse Fourier transform of a particular characteristic function that is evaluated using standard quadrature routines.

The paper is organized as follows - section 2 discusses the general equilibrium setting and solves the social planner's problem, section 3 discusses asset pricing implications and derive option prices, section 4 discusses our MCMC algorithm and presents the empirical findings and section 5 concludes.

2 The Model

In this section, we define our general equilibrium framework in continuous time and derive the consumption-to-wealth ratio which is at the heart of our analysis. We start by specifying the underlying preferences that the agent is endowed with and dynamics that he is exposed to. Then, we solve for optimal consumption of the representative agent. In the following section, we derive the agent's marginal utility process and solve for equilibrium risk-prices under the given preferences, dynamics and optimal consumption plan. Finally, we use this marginal utility process to price put options under risk-adjusted dynamics.

The representative agent is infinitely lived and is endowed with the stochastic differential utility (SDU) of Duffie and Epstein (1992), which is the continuous-time version of Epstein and Zin (1989). The aggregator, $f(C, J)$, can be represented directly under standard normalization by

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right] \tag{1}$$

where C is the current consumption and J is the continuation utility given by the agent's value function. The parameters of this utility function are β - the discount rate for this infinitely lived agent, ψ - the Intertemporal Elasticity of Substitution (IES) and γ - risk-aversion. Unlike time-separable CRRA utility functions, SDU separates the close link between risk-aversion and intertemporal substitution. Under time-separable CRRA preferences, $\gamma = \frac{1}{\psi}$, whereas under SDU that tight relationship is broken. In fact, Schroder (1999) shows that agents prefer early (late) resolution of uncertainty if $\gamma > 1/\psi$ ($\gamma < 1/\psi$).

2.1 Prices and Dividends

The basic set-up of our model is the full information economy of Ai (2009). The asset market contains a default-free bond whose price is Z_t and it earns a risk-free rate r_t^f. There is a stock whose price is P_t and it is a claim to the firm that converts capital into consumption good. Any other asset, like options, is available in zero-net supply.

Dividend flows from the firm are paid in terms of aggregate consumption that is represented by C_t. In time period dt, the firm produces consumption flows $C_t dt$. Aggregate wealth, R_t, is characterized by the equilibrium return on stock plus the dividend-yield every period, $\frac{dR_t}{R_t} = \frac{dP_t}{P_t} + \frac{C_t}{P_t}dt$.

2.2 Consumption and Capital

The state of the economy is governed by a one-dimensional process θ_t which is the expected return on the production technology and captures the state of the economy. Capital grows linearly using a linear $A(\theta_t)K_t$ production function with stochastic production fluctuations $A(\theta_t)$, where θ_t is the expected return on the production technology. θ_t follows a mean-reverting Ornstein-Uhlenbeck process of the form

$$d\theta_t = \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta \tag{2}$$

Here, the stochastic production fluctuation $A(\theta_t)$ represents a production shock that is net of depreciation. Capital growth will be subjected to the same stochastic fluctuation of the production shock - one due to a transient shock with volatility σ_K and the other due to the mean-reverting stochastic component θ_t. Given the linear specification of the production function, capital formation in this economy is given by the usual law of motion[3]

$$\begin{aligned} dK_t &= A(\theta_t)K_t - C_t dt \\ &= [\theta_t dt + \sigma_K dB_K]K_t - C_t dt \end{aligned}$$

In this economy, dB_K and dB_θ are correlated Brownian motion shocks with correlation $\rho > 0$. σ_K and σ_θ represent volatility of iid shocks to production and returns to production respectively. Shocks to the production function are due to a transient shock with volatility σ_K and a growth rate shock due to θ_t which defines the state of the economy. The process of θ_t is known to the representative agent, unlike in Ai (2009), where the agent has uncertainty over this shock. The stationary distribution of the shocks, θ_t, is determined by $\theta_t \sim N(\bar{\theta}, \frac{\sigma_\theta}{\sqrt{2\delta}})$. The time-series property of risk-premia that we derive endogenously is captured by the sign and distance of the current state θ_t from the average growth rate $\bar{\theta}$. We define high (low) growth rate periods when the current state, θ_t, is above (below) $\bar{\theta}$. Growth rate mean-reverts around $\bar{\theta}$ with the speed of mean-reversion given by δ.

Since capital is the only factor of production, the return on the stock is the same as the return on capital. Thus, $\frac{dP^*}{P^*} = \frac{dK^*}{K^*}$, and if $P_0 = K_0$ then $P_t^* = K_t^*$. Hence, return on aggregate wealth follows

$$\frac{dR_t^*}{R_t^*} = \frac{dP_t^*}{P_t^*} + \frac{C_t^*}{P_t^*}dt \tag{3}$$

$$= \frac{dK_t^*}{K_t^*} + \frac{C_t^*}{K_t^*}dt \tag{4}$$

[3] Ai (2009) shows this is the same as the traditional law of motion of capital that is standard in real business cycle models. Let $a_{t,\Delta}$ denote the Total Factor of Productivity in the infinitesimal time period $[t, t+\Delta]$. If I_t be investment rate, then $K_{t+\Delta} = (1 - d_{t,\Delta})K_t + I_t\Delta$, where $d_{t,\Delta}$ is the depreciation rate during the same time interval. The usual resource constraint is $I_t\Delta + C_t\Delta = a_{t,\Delta}K_t$. Substituting in to the capital law of motion produces $K_{t+\Delta} - K_t = (a_{t,\Delta} - d_{t,\Delta})K_t - C_t\Delta$. Redifining $A_{t,\Delta} = a_{t,\Delta} - d_{t,\Delta}$ and taking limits produces $dK_t = A(\theta_t)K_t - C_t dt$ where $A(\theta_t)$ is the linear production shock that is defined to be $\theta_t dt + \sigma_K dB_K$. In other words, the production technology has a stochastic growth rate that is autocorrelated and an iid shock.

Put option prices on aggregate wealth level R_t^* will serve as our proxy for price of insurance claim in this economy. Next we derive equilibrium consumption and the consumption-to-capital ratio in this economy.

2.3 Social planner's problem and Equilibrium Consumption

The First and Second Welfare Theorems apply directly, and thus competitive equilibrium allocations coincide with the Pareto allocations of the social planner. The social planner's Pareto optimality problem is

$$J(K_t, \theta_t) = \sup_{C_t} E_t \left[\int_t^\infty f(C_s, J_s) ds \right] \tag{5}$$

subject to

$$\begin{aligned} dK_t &= K_t[\theta_t dt + \sigma_K dB_K] - C_t dt & (6) \\ d\theta_t &= \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta & (7) \end{aligned}$$

Propostion 1: The solution of the social planner's problem (5) is given by

$$J(K_t, \theta_t) = H(\theta_t) \frac{K_t^{1-\gamma}}{1-\gamma} \tag{8}$$

Optimal capital is given by

$$\frac{dK_t^*}{K_t^*} = \mu_K(\theta_t)dt + \sigma_K dB_K \tag{9}$$

where

$$\mu_K(\theta_t) = \left[\theta_t - \beta^\psi H(\theta_t)^{\frac{1-\psi}{1-\gamma}} \right]$$

and optimal consumption, $C_t^* = y^* = \beta^\psi H(\theta_t)^{\frac{1-\psi}{1-\gamma}} K_t$ where $H(\theta_t)$ satisfies a functional relationship expressed in terms of a second order ODE given in (27). Aggregate stock price follows $\frac{dP_t^*}{P_t^*} = \frac{dK_t^*}{K_t^*}$ in (9) and thus the total returns to aggregate wealth is

$$\frac{dR_t^*}{R_t^*} = \frac{dP_t^*}{P_t^*} + \frac{C_t^*}{P_t^*} dt = \theta_t dt + \sigma_K dB_K \tag{10}$$

Proof: See Appendix A.

Return on the aggregate wealth follows the same process as the production fluctation shock $A(\theta_t)$. This simplified result is due to our choice of a linear production function. If the expected return on the production technology, θ_t, is higher (lower) than $\bar{\theta}$, that means expected growth rate of the economy is higher (lower) than the average growth rate, which implies aggregate wealth is expected to increase (decrease) at a higher (lower) than average rate.

Going forward, we assume that we are only dealing with optimal quantities given by Proposition 1 and hence we drop the "*" super-scripts. For $\psi < 1$, we find a particularly convenient solution of the value function of the form

$$J(K_t, \theta_t) = \frac{K_t^{1-\gamma}}{1-\gamma} H(\theta_t) = \frac{K_t^{1-\gamma}}{1-\gamma} e^{\left(\tilde{a} + \tilde{b}\theta_t + \frac{1}{2}\tilde{c}\theta_t^2\right)} \tag{11}$$

where $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ are functions of the underlying parameter space given in the Appendix. Now, whether the value function is increasing or decreasing in the underlying state will explicitly depend on the state θ_t.

The key asset pricing results in this paper is driven by the endogenously determined consumption to capital ratio given by $\frac{C_t^*}{K_t} = \beta^\psi H(\theta_t)^{\frac{1-\psi}{1-\gamma}}$, where $H(\theta_t)$ is exponentially quadratic in θ_t for $\psi < 1$. The dynamic properties of risk-premia that follow from this relationship are due to both SDU and time-varying growth rates. This result is distinct from Ai (2009) who only considers an exponentially affine solution of the value function, which we obtain as a special case for $\psi \geq 1$. In fact, for $\psi \geq 1$, we get the same solution of the value function that corresponds to the full information economy of Ai (2009). Moreover, we also extend the results derived in Weil (1990) who considers an iid economy. To determine the counter-cyclical nature of risk-premia that drives demand for insurance, let's first analyze the consumption/savings propensity of the agent. We only provide intuition in this section and tie-in the underlying observations with income and substitution effects in the asset pricing section where we derive risk-free rates and risk-premia.

First, observe that when $\psi = 1$, the consumption to capital ratio is a constant and independent of the underlying state or the risk-aversion parameter. This confirms the finding of Weil (1990) that the "myopia" in the consumption and savings decisions exhibited by traditional Von Neumann-Morgenstern utility functions stem from unit IES and not from unit coefficient of risk-aversion. In order to judge the agent's propensity to save or consume, let's focus on how the optimal consumption to capital ratio changes when the growth rate changes. In our case, whether or not the consumption to capital ratio is increasing or decreasing in the underlying state depends not only on the agent's risk-aversion or IES, but also on the underlying state as is evident from the exponentially quadratic solution of the value function $H(\theta_t)$.

$$\begin{aligned} \frac{1}{\frac{C}{K}} \frac{\partial \frac{C}{K}}{\partial \theta_t} &= \frac{1-\psi}{1-\gamma} \frac{H'(\theta_t)}{H(\theta_t)} \\ &= \frac{1-\psi}{1-\gamma} \left(\frac{\gamma - 1}{\delta} (1 + \tilde{c} \sigma_K \sigma_\theta \rho) + \tilde{c}(\theta_t - \bar{\theta}) \right) \end{aligned} \tag{12}$$

where $\tilde{c} > 0$ is shown in the Appendix. This solution holds for $\psi < 1$, and assume that the agent has $\gamma > 1$. Since $\psi < 1$, heuristically that suggests that the agent is averse to substituting consumption across time. We further assume that γ is big enough such that $\gamma > \frac{1}{\psi}$ and the agent prefers early resolution of uncertainty. With our choice of the preference parameters, the leading term of (12), $\frac{1-\psi}{1-\gamma}$, is always negative. Once we fix the preference parameters, the propensity to consume or save depends on whether current growth rate θ_t is higher or lower than the average growth rate $\bar{\theta}$.

Assume the economy receives a positive shock such that the growth rate increases. First, let us consider this positive shock in a good state of the economy where $\theta_t > \bar{\theta}$. Therefore, the consumption to capital ratio decreases and the savings motivation prevails. This shows satiation for consumption in high growth rate state of the economy such that if growth rate increases even more, the agent will save more relative to consumption even with IES less than one. Now, assume that the economy receives the same positive shock in a bad state of the economy where $\theta_t < \bar{\theta}$. Clearly, the agent is less satiated in the bad state and the propensity to consume is higher than savings.[4] With a positive shock in the bad state, the agent infers high growth rates and high level of consumption in the future. Now, the agent with low IES will show less desire to substitute consumption intertemporally and increases consumption now. The result shows that what drives the desire to substitute consumption intertemporally is IES less than one in combination with underlying state. In high growth rate periods, the agent feels satiated and responds to a positive shock by increasing savings relative to consumption. In low growth rate periods, the agent's low desire to substitute consumption kicks in and he responds to a positive shock by increasing consumption.

Now, assume the economy receives a negative shock in the growth rate. In good state of the economy ($\theta_t > \bar{\theta}$), if the economy receives a negative shock, then the agent's consumption motivation will dominate. The agent foresees lower growth rates in the future, his low desire to substitute consumption into the future prevails and the agent increases consumption today. However, the same negative shock in a low growth state ($\theta_t < \bar{\theta}$) has

[4]The sign of the derivative is negative for $\theta_t < \bar{\theta}$ provided the stochastic term inside the paranthesis in (12) is negative enough to overcome the constant term. For small risk-aversion parameters and less autocorrelated states (high δ) the constant term is small relative to the stochastic term.

very different consequences. Now, the growth rate is not only poor but the economy is headed for the worse. Even if the agent has low IES, he will give up consumption and save more. In high growth rate states, the agent interprets a negative shock very differently than in low growth states. In the former, the agent reacts to decreasing growth rates with higher consumption because the growth rate of the economy is still high. In the latter, the agent increases savings because the economy is headed towards a disastrously bad state.

The differential actions of the agent in response to the changes in the growth rate of the economy can also be looked at through mean-reversion of the growth rates. When growth rates start to mean-revert to $\bar{\theta}$ from either good times ($\theta_t > \bar{\theta}$) or bad times ($\theta_t < \bar{\theta}$), the agent's consumption motivation is higher than his propensity to save. However, as growth rates move away from $\bar{\theta}$, the propensity to save is higher. The heuristic notion that an agent with low IES is averse to substituting consumption over time holds true only when growth rates revert towards the average growth rate, but the agent is willing to save more when growth rates diverge away from the average growth rate. In the next section, we will rationalize this phenonmenon by differential actions of the agent in response to changes in the certainty equivalent interest rate and the volatility of marginal utilities that gives rise to stochastic risk-premia. As Weil (1990) points out, agents who prefer early resolution of uncertainty are exposed to certainty equivalent fluctuations of utility over time which are of large amplitude. As such, there is always a tradeoff between "safety" and "stability" of utility. We will see the instability of the utility, captured by stochastic volatility of marginal utility, can be high or low across good or bad times which gives rise to time-variation in prices of risk. At the same time, the certainty equivalent interest rate, changes which dictates the agent's income or substitution effects.

For $\psi > 1$ and $\gamma > 1$, the Appendix shows that the exponentially linear solution of $H(\theta_t)$ guarantees that the consumption to capital ratio is strictly decreasing in θ_t. Therefore, an agent responds to increasing (decreasing) growth rates by saving (consuming) more. We will show that this is where the substitution effect always dominates the income effect, which is consistent for an agent who has high IES.

Before we move onto risk-premia, let us first consider the endogenously determined consumption growth process which identifies the source of time-variation in risk-premia.

$$\frac{dC^*}{C^*} = \mu_C(\theta_t)dt + \sigma_K dB_K + \frac{1-\psi}{1-\gamma}\frac{H'(\theta_t)}{H(\theta_t)}\sigma_\theta dB_\theta \tag{13}$$

where

$$\mu_C(\theta_t) = \mu_K(\theta_t) + \frac{1-\psi}{1-\gamma}\frac{H'(\theta_t)}{H(\theta_t)}\delta(\bar{\theta}-\theta_t) + \frac{1-\psi}{1-\gamma}\left[\frac{H''(\theta_t)}{H(\theta_t)} + \frac{H'(\theta_t)^2}{H(\theta_t)^2}\frac{\gamma-\psi}{1-\gamma}\right] + \frac{1-\psi}{1-\gamma}\frac{H'(\theta_t)}{H(\theta_t)}\sigma_K\sigma_\theta\rho \tag{14}$$

and for $\psi < 1$

$$\frac{H'(\theta_t)}{H(\theta_t)} = \left(\frac{\gamma - 1}{\delta}(1 + \tilde{c}\sigma_K\sigma_\theta\rho) + \tilde{c}(\theta_t - \bar{\theta})\right)$$

Notice, if $\psi = 1$, then consumption growth and capital growth would be the same and they will both be linear in θ_t, but that is not the case if $\psi \neq 1$. If $\psi \neq 1$, expected consumption growth rate, $\mu_C(\theta_t)$, is non-linear in the expected growth rate. More interestingly, notice that volatility of consumption growth is now stochastic. If $\theta_t > \bar{\theta}$, then $\frac{H'(\theta_t)}{H(\theta_t)} > 0$ and with $\psi < 1$ and $\gamma > 1$, the volatility due to the stochastic growth rate, $\frac{1-\psi}{1-\gamma}\frac{H'(\theta_t)}{H(\theta_t)}\sigma_\theta$, is now negative which reduces total volatility of consumption growth in good times. But, if $\theta_t < \bar{\theta}$, then $\frac{H'(\theta_t)}{H(\theta_t)} < 0$,[5] which implies that with $\psi < 1$ and $\gamma > 1$, overall consumption volatility will be high. This stochastic volatility of consumption growth that is high during bad times and low during good times is responsible for corresponding stochastic volatility of marginal utilities whose variation is substantial across good or bad times.

Notice, the stochastic volatility of consumption growth is determined endogenously. The current literature (Benzoni, Collin-Dufresne and Goldstein (2005), Drechsler and Yaron (2007), Shaliastovich (2008)[6]) specifies it exogenously, but here we obtain it endogenously through preferences and deeper economic shocks. This stochastic volatility component generates time-varying risk-premia. Thus, in models where stochastic volatility of consumption growth is specified exogenously, time-varying risk-premia is obtained immediately, whereas we develop this phenomenon within our model setting. As have been argued earlier, for $\psi > 1$, $H(\theta_t)$ is exponentially linear in θ_t. As such, $\frac{H'(\theta_t)}{H(\theta_t)}$ is a constant and the stochastic volatility of consumption growth disappears.

The above analysis crucially depends on $\tilde{c} \neq 0$ in the solution of the value function. We obtain a solution of the value function $H(\theta_t)$ by adopting a linearization scheme from Campbell, et. al. (2004) that is an approximation of the value function around the unconditional mean of the optimal consumption to capital ratio, i.e. we look for a particular solution of the value function around the point $\mu = E\left[\ln\left(\frac{C^*}{K}\right)\right]$. A solution of $H(\theta_t)$ is proposed that is exponentially quadratic in θ_t. The quadratic has two roots and one of them is zero. Figure 2 shows that the zero-root is a "better" solution of the value function when $\psi \geq 1$. However, the non-zero root is a "better" solution when $\psi < 1$. By "better", we imply that plugging in the zero or non-zero root solution of $H(\theta_t)$ to the left-hand side of the functional relationship that $H(\theta_t)$ satisfies in (27), the non-zero root is closer to the right-hand side of (27), i.e. 0, when $\psi < 1$, but the zero-root is closer to the right-hand side when $\psi \geq 1$. Taking parameter values that are estimated from option prices, we plot

[5] as long as the stochastic term is significantly negative enough to overcome the constant term

[6] The effect of stochastic volatility of consumption is small in Shaliastovich (2008).

the left-hand side of (27) with zero and non-zero root solutions against the growth rates θ_t and different values of ψ in Figure 2. We confirm that under the parameter values of our system, the exponentially quadratic solution of $H(\theta_t)$ is a better approximation of the solution of (27) for $\psi < 1$. The Appendix also shows that as ψ approaches one, we recover the well-known exponentially affine solution by simply plugging in the zero-root. Furthermore as γ approaches one, we recover the log utility case from the zero-root solution. We proceed with the assumption, that $\psi < 1$, and the non-zero quadratic root gives the solution of the value function. To see how the above setting translates into time-varying risk-premia, let's now focus on the asset-pricing implications.

3 Asset Pricing

In order to price assets in this setting, we need to compute the marginal utility of an agent with preferences and dynamics from above. We derive the price of risk of the agent in this section and in the next section price put options which will be our proxy for price of insurance in this economy. We price put options on aggregate wealth from Proposition 1 given by

$$\frac{dR_t}{R_t} = \theta_t dt + \sigma_K dB_K$$

because the intuition for insurance motivation on aggregate wealth is very simple.

To show counter-cyclicality in insurance motivation, we focus on how an agent responds to an adverse shock to expected wealth in good or bad times. In good times, when $\theta_t > \bar{\theta}$, the agent's wealth is expected to increase at an above average growth rate. We showed that in response to a negative shock in growth rates in good times, the agent's propensity to consume still prevails. Clearly, the agent's willingness to pay for insurance is low. However, when the agent faces a negative shock to growth rate in bad states, when $\theta_t < \bar{\theta}$, the agent's demand for insurance on his wealth is high. This shows heuristically why we expect put option prices to be higher in bad times over good times. Notice, the volatility of the "underlying" wealth process is constant which is a result of the stylized economy we consider here. We now show the nature of prices of risk in this economy that drives the differential demand of put options across good and bad times.

Proposition 2: Given equilbirum consumption in (13) from above and the utility function

(1), the dynamics of the marginal utility process, Λ_t, of the investor with $\psi < 1$ is given by

$$\begin{aligned}
\frac{d\Lambda_t}{\Lambda_t} &= \frac{df_C}{f_C} + f_J dt \\
&= -r_t^f - \gamma\sigma_K dB_K + \frac{H'(\theta)}{H(\theta)}\sigma_\theta dB_\theta \\
&= -r_t^f - \gamma\sigma_K dB_K - \left(\frac{1-\gamma}{\delta}(1+\tilde{c}\sigma_K\sigma_\theta\rho) + \tilde{c}(\bar{\theta}-\theta_t)\right)\sigma_\theta dB_\theta \qquad (15)
\end{aligned}$$

where r_t^f is the risk-free rate and is equal to

$$r_t^f = \left(\frac{\gamma-1}{\delta}(1+\tilde{c}\sigma_K\sigma_\theta\rho)\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2\right) + \theta_t + \tilde{c}\sigma_K\sigma_\theta\rho(\theta_t-\bar{\theta}) \qquad (16)$$

and the equilibrium risk premia is given by

$$\begin{aligned}
-Cov\left(\frac{d\Lambda}{\Lambda}, \frac{dP}{P}\right) &= -Cov\left(\frac{d\Lambda}{\Lambda}, \frac{dK^*}{K^*}\right) \\
&= \gamma\sigma_K^2 + \left(\frac{1-\gamma}{\delta}(1+\tilde{c}\sigma_K\sigma_\theta\rho) + \tilde{c}(\bar{\theta}-\theta_t)\right)\sigma_K\sigma_\theta\rho \qquad (17)
\end{aligned}$$

Proof: See Appendix B.

We can understand the time-variation in the consumption to capital ratio from the previous section through the dynamics of the risk-free rate. The risk-free rate is procyclical and increases (decreases) when the underlying growth rate increases (decreases). Going forward, we define the average risk-free rate as the risk-free rate when the underlying growth rate is the average growth rate, i.e. $\theta_t = \bar{\theta}$. We observed in the previous section that when growth rates are higher than average growth rate, $\theta_t > \bar{\theta}$, the consumption motivation is stronger if growth rates are decreasing but the savings motivation dominates if growth rates increase. When growth rates are higher than the average growth rate, the risk-free rate is higher relative to the average risk-free rate. If the risk-free rate is decreasing when times are good (i.e. when the price of certainty equivalence is increasing in good times), the agent's propensity to consume is higher. If the risk-free rate is increasing when times are good (i.e. when the price of certainty equivalence is decreasing), the agent's propensity to save is higher. They are both evidence that the substitution effect dominates the income effect when growth rates are higher than the average growth rate.

Similarly, when growth rates are lower than the average growth rate, the savings motivation dominates if growth rates are decreasing and the propensity to consume prevails if growth rates are increasing. In these states, if the risk-free rate is decreasing (i.e. the price of certainty equivalence increases), the agent saves more. Likewise, if the risk-free rate is increasing, the agent consumes more. Thus, when growth rates are less than the average growth rate, the income effect dominates the substitution effect. Thus, the time-variation of the consumption to capital ratio that we observed earlier is due to fluctuating income and substitution effect across good or bad states. When growth rates are higher (lower) than the average growth rate, the substitution (income) effect dominates the income (substitution) effect.

Notice, once we fixed $\gamma > 1$, the above time-variation in income and substitution effect is solely due to $\psi < 1$. We saw earlier that if $\psi > 1$ and $\gamma > 1$, then the exponentially affine solution of $H(\theta_t)$ ensures that $\tilde{c} = 0$. In this case, the consumption to capital ratio is always decreasing as θ_t increases. Therefore, as risk-free rate increases and the price of certainty equivalence decreases, the agent saves more. Thus, for $\psi > 1$ and $\gamma > 1$, the substitution effect always dominates the income effect.

The $\tilde{c}$ term, which comes from the loading on the quadratic coefficient in the value function, creates lower (higher) risk-free rate when growth rates are lower (higher) than the average growth rate, $\bar{\theta}$. Thus, this plays a valuable role in generating stronger precautionary savings motive during low growth rate states when $\theta_t < \bar{\theta}$. Similarly, during high growth rate states, when $\theta_t > \bar{\theta}$, the $\tilde{c}$ term drives down risk-free rates to the point that substitution effect starts dominating the income effect and precautionary savings is muted. There are two other precautionary savings terms - $\gamma\sigma_K^2$ and θ_t both of which can drive down the risk-free rate. The first term comes from standard power utility settings, while the second one is due to stochastic growth rates. If $\psi \geq 1$, these are the only precautionary savings term and the time-variation in the risk-free rate is solely due to θ_t. If $\psi < 1$, the quadratic component of the value function introduces the additional $\tilde{c}(\theta_t - \bar{\theta})$ term that greatly maginifies the precautionary motive in bad times, and mutes precautionary savings in good times.

Moving onto risk-premia, we are able to generate counter-cyclical risk-premia based on time-variation in the income and substitution effects. The first term in the expression for risk-premia in this economy given by (17) is due to the compensation for being exposed to the iid shock of capital growth and is a constant. The second risk-premia term is compensation due to the time-varying growth rate. It is counter-cyclical and reflects how an agent with SDU interprets a shock in the growth rate in good or bad times. The income effect that we see when growth rates are below average, i.e. $\theta_t < \bar{\theta}$, is simply a precautionary savings effect. When precautionary savings motive dominates under poor growth rates, the agent's risk-premia is high as is confirmed through (17). Similarly, in good times when $\theta_t > \bar{\theta}$, the substitution effect now starts to dominate the income effect which implies that precautionary savings motive dissipates. Naturally the risk-premia from exposure to growth rate shocks is low. Notice that the counter-cyclical component of the risk-premia is due to $\psi < 1$. If $\psi > 1$, then the value function is exponentially affine making $\tilde{c}$ zero and risk-premia constant.

The second risk-premia term due to SDU and stochastic growth rates has been termed long-run risk in the literature. The first component - $\frac{1-\gamma}{\delta}(1 + \tilde{c}\sigma_K\sigma_\theta\rho)$ is constant and vanishes as $\gamma \to 1$. Unlike traditional risk-premia components, this actually decreases with higher risk-aversion. This component is inversely proportional to δ and is magnified for strongly autocorrelated states. The stochastic component of this premia - $\tilde{c}(\bar{\theta} - \theta_t)$ is entirely due to $\psi < 1$ and does not depend on risk-aversion, γ. This component is magnified for less autocorrelated states since $\tilde{c} \propto \delta$. It generates high risk-premia in low growth rate states when $\theta_t < \bar{\theta}$ and low (even negative) risk-premia in high growth rate states when $\theta_t > \bar{\theta}$. This counter-cyclical risk-premia is what we need in order to generate the counter-cyclicality in the demand for put option insurance. The strong precautionary savings motivation in bad states generates higher price for insurance in these periods. In good times when the precautionary savings motive dissipates and substitution effect takes over, the resulting risk-premia is low generating low prices for put options. Clearly, when we emphasize the role of the stochastic component of risk-premia, we require the underlying states to be far less autocorrelated. The "long-run" risk in our setting that generates high time-variation in risk-premia is thus due to short-run dynamics.

The results above also tie in what Weil (1990) observed about agents who prefer early resolution. Weil (1990) points out that agents who seek early early resolution of uncertainty typically trade-off between "safety" and "stability" of utility. As the income and substitution effects change across the business cycle, the added precautionary savings term changes drastically altering his preference for certainty equivalence and prices of risk. In bad times, the agent's high demand for certainty equivalance lowers the risk-free rate and generates high stochastic volatility in his marginal utilities which raises risk-premia. In good times, the precautionary savings motive dissipates and the agent feels less risk-averse which also simultaneously lowers his preference for certainty equivalence thus raising the risk-free rate and lowers risk-premia.

This time-variation in risk-premia under SDU is vastly different from what is in the long-run risk literature. Typical time-variation in long-run risk is due to exogenous specification of stochastic volatility of consumption growth, which goes back to Bansal and Yaron (2004). Here, we derive stochastic volatility of consumption growth and time-varying risk-premia jointly from exploring non-linearities in the log consumption-to-capital ratio. The long-run risk model here is also very close to the habit formation models due to Campbell and Cochrane (1999) and Santos and Veronesi (2008). Whereas the dynamics of the habit formation models are driven by what is termed surplus consumption ratio, here the model dynamics are due to fluctuating growth rates of the economy. Like Campbell and Cochrane (1999), our risk-free rate contains additional precautionary savings terms. They enforce restrictions at this point in order to make the risk-free rate constant, but we do not impose any such restriction that could be justified within our general equilibrium framework. Thus, we are close to the habit formation set-up of Santos and Veronesi (2008). Like Santos and Veronesi (2008), the volatility in the risk-free rate that we generate is also very high.

If $\psi \geq 1$, the stochastic component vanishes for exponentially affine $H(\theta_t)$ and now the risk-premia is constant - $\frac{\gamma-1}{h_1+\delta}$. The risk-premia is increasing in risk-aversion, strongly autocorrelated states and low long-run average of consumption to capital ratio. At $\psi = 1$, the consumption to capital ratio is a constant - β and thus the long-run risk-premia becomes $\frac{\gamma-1}{\beta+\delta}$.

Since we have already generated counter-cyclical risk-premia which determines an agent's willingness to insure himself against an adverse shock, what remains to be shown is whether the time-variation in risk-premia is high enough to generate the prices of put option insurance that we observe in the economy. In other words, the key question is whether the additional precautionary savings motive that we generate in bad times is sufficient to produce high price for put option insurance. Whether or not we can generate that from parameters that are reasonable is the quantitative aspect of the theory. We leave it for the empirical section where we actually estimate the parameters from the consumption dynamics and option prices. The next section shows how we price such put options.

3.1 Options

Other assets in the economy, as long as they are in zero net supply, will also be priced in this equilibrium. A put option on aggregate wealth that displays the insurance motivation of the agent is one such asset. Before we proceed to formal pricing, it is instructive to analyze what generates state-dependence in the prices of such assets. Clearly, the level of the stock and the strike-price introduces state dependence, but they enter into the pricing relationship as a ratio creating what is termed as "moneyness" in the literature that characterizes the cross-section of options. The counter-cyclicality of risk-premia due to stochastic growth rates is responsible for generating the counter-cyclical prices of put option insurance. It enters the pricing relationship through the risk-neutral probability that the option finishes in the money and generates a payout. The reduced form option pricing literature introduces state-dependence in the risk-neutral probability (e.g. Heston (1993)) through exogenous specification of stochastic volatility in the underlying asset, whereas in ours it is generated endogenously due to the stochastic volatility of the marginal utility process. In our simple set-up, the underlying wealth process on which options are written has a constant volatility, but the stochastic volatility of the marginal utility process (i.e. the pricing kernel) generates the state-dependence in risk-neutral probability. Therefore, from looking at the prices of these contracts, we can infer the underlying economic state.

Let's consider an European put option written on aggregate wealth, R_t. It has a payoff of $\max[X - R_T, 0]$ at time T. In this case, the put option insures the agent from a loss in the wealth level below X at time T. As such, this put option offers *early resolution of uncertainty* regarding the agent's wealth at time T. We know that agents with SDU and $\gamma > \frac{1}{\psi}$ will exhibit preference for early resolution of uncertainty, and thus this is a natural setting to study prices of these contracts that offer precisely that. First, we price the corresponding call option and then use put-call parity to back out the put option price. If the call option has a strike price of X, then it pays $\max[R_T - X, 0]$ unit of wealth at maturity T.

Its price at time t, $G(R_t, \theta_t, \tau)$, satisfies

$$E_t[dG] = r_t^f G - GE_t\left[\frac{d\Lambda}{\Lambda}\frac{dG}{G}\right] \tag{18}$$

where $\tau = T - t$. The first term is the familiar risk-neutrality term and the second term is the adjustment for the risk. The adjustment for risk comes from the pricing kernel in the previous section which reflects adjustment for the transient component and stochastic long-run risk premia.

Since the dynamics here is completely affine the method of Duffie, Pan and Singleton (2000) will directly apply and the semi-closed solution for option prices is given by

Proposition 3: The price of a call option that pays $\max[R_T - X, 0]$ at maturity is given by

$$G(R_t, \theta_t, \tau) = R_t P1(g_t, \theta_t, \tau) - XP2(g_t, \theta_t, \tau) \tag{19}$$

where $g_t = \ln(R_t)$ and $P1$ satisfies restriction (45). $P2$ satisfies an exactly analogous restriction also with a set of ODE's given by (46).

Proof: See Appendix C.

Relative option prices can be obtained by dividing through the option price by R_t. Using put-call parity, the corresponding relative put option price is

$$\frac{\text{Put}}{R_t} = \frac{X}{R_t}(Z(\theta_t, \tau) - P2(g_t, \theta_t, \tau)) - (1 - P1(g_t, \theta_t, \tau)) \tag{20}$$

where $Z(\theta_t, \tau)$[7] is the price of a zero-coupon bond that matures in τ-period and is endoge-

[7] In the emprical section, since we are only going to look at very short-maturity options $Z(\theta_t, \tau)$ should

nously determined given the risk-free rate r_t^f. The solution of $P1$ and $P2$ reveals that pure levels of R_t or X enter into the option price formula (20) through the ratio X/R_t which has an interpretation of its own called moneyness. If $X/R_t = 1$, then the put option is at-the-money and the cost of the put option is the insurance that the agent is willing to pay such that his wealth level at some point in the future T doesn't fall below the current wealth level at time t. If $X/R_t = 0.9$, then the put option protects the agent from more than a 10% drop in wealth level, i.e. it is 10% out-of-the-money (OTM). Note, the above put option formula also depends on the underlying parameter space of the economy, and is suppressed here for brevity.

For call options, $P1$ has to satisfy the boundary condition that $P1(g_T, \theta_T, 0) = 1_{\{g_T \geq \ln(X_T)\}}$. Same holds true for $P2$. As such, $P1$ or $P2$ is the "risk-adjusted" probability that the call option finishes in the money. Likewise, $1 - P1$[8] $(1 - P2)$ is the "risk-adjusted" probability that the put option finishes in the money. Naturally, since the expected growth rate of wealth is time-varying, the probability that it is below a particular level X is also time-varying. However, the compensation for risk of this expected growth rate is also stochastic and the risk-prices of the agent show up prominently in $P1$ or $P2$. Thus, the above "risk-adjusted" probabilities that are priced into put options reflect underlying stochastic risk preferences in this economy through the ODEs that satisfy $P1$ and $P2$ in (43) and (46). Low growth rates reflect not only a higher probability of wealth loss but these are also high risk-premia states of the world. Thus, any contract to insure wealth level in bad times will have a higher probability of a payout under the "risk-adjusted" probability and hence more expensive. Good times, on the other hand, not only signal increase in wealth but also low risk-premia states where the "risk-adjusted" probability of a put option payout is low and that drives down prices of these contracts. Hence, the equilibrium setting implies that the time-variation in put option prices is due to time-variation in long-run risk from SDU.

Note, the time-series property is coming purely from the underlying growth rates because once the current wealth level is converted into moneyness, X/R, it starts to characterize the cross-section of put options at time t. Given a level of moneyness at time t, $1 - P1$ $(1 - P2)$ reflects the "risk-adjusted" probability that the wealth level will fall by at least X/R of its current level and the price of put option reveals how much the agent is willing to spend to insure against that loss. As such, X/R - moneyness, represents the cross-section of put options in this context.

Thus, we generate general equilibrium put option prices consistent with SDU, $\psi < 1$,

be very close to 1.

[8]Notice that since we will consider only short maturity option the price of the zero-coupon bond, $Z(\theta_t, \tau)$ is very close to 1, and $Z(\theta_t, \tau) - P1 \approx 1 - P1$

risk-aversion greater than 1, and time-varying production fluctuations. The risk-premia that is priced into the option prices reflect the agent's strong precautionary savings motivation in poor economic states and high satiation in good economic states. From an empirical stand-point, the above presents two challenges. First of all, since the underlying economic states, θ_t, are unobservable, we need to filter them out from a time-series of option prices to compute the model implied risk-premia in (17) implied by put option prices. The put option prices in this general equilibrium framework have put severe non-linear restrictions in the entire parameter space that governs preferences, shocks and evolution of growth rates. The empirical challenge is to estimate parameters and state variable from this complicated state-space dynamics by preserving all the non-linearities induced by the theory. Furthermore, since we undertake a general equilibrium approach, we also address aggregate consumption growth derived in (13). Together, they will estimate a complete general equilibrium state-space characterized by non-linear cross-equation and cross-quantity restrictions, across both macroeconomic quantities and asset prices. The following section outlines this methodology.

4 Empirical Methodology and Results

The general equilibrium setting described above creates the following state-space model. Equilibrium consumption and relative put option prices can be written jointly as

$$\frac{dC^*}{C^*} = \mu_C(\theta_t, \lambda)dt + \sigma_K dB_K + \frac{1-\psi}{1-\gamma}(\tilde{b} + \tilde{c}\theta_t)\sigma_\theta dB_\theta \quad (21)$$

$$\frac{Put}{R_t} = \frac{X}{R}(Z(\theta_t, \tau) - P2(g_t, \theta_t, \tau)) - (1 - P1(g_t, \theta_t, \tau)) \quad (22)$$

$$d\theta_t = \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta \quad (23)$$

where $\lambda = \{\delta, \bar{\theta}, \sigma_K, \sigma_\theta, \rho, \gamma, \psi, \beta, \mu\}$ is the parameter space of the underlying economy.

The first two equations are derived endogenously in this economy and they are functions of the underlying dynamics and preferences. The parameters that govern the system, λ, are embedded in every term on the right hand side of the first two equations along with the expected growth rate θ_t. Jointly, they define a state-space system where the underlying state, θ_t, shows up non-linearly in the equilibrium quantities. Our goal in the empirical exercise is to obtain the parameter estimates of λ along with time-series estimates of θ_t. In addition, we are interested in the model implied put option prices obtained from our estimation and equation (22).

In our empirical analysis we approximate the system with an Euler discretization and add an iid $N(0, \sigma^2)$ error term to the put equation (22). This gives us a discrete non-linear state-space model.

4.1 Data

We use consumption data of Real Non-durables and Services obtained from the Bureau of Labor Statistics in annual frequency from 1929-2006. We divide the sum of Non-durables and Services by a population series to make it Per Capita Real Consumption of Non-durables and Services. This is the longest time-series of Real Consumption growth that is available in the US time-series. On the other hand, traded options data on the S&P 500 Futures Index is available daily from 1988-2006[9] and are interpolated at regular intervals of strike prices.[10] A full panel of option prices (both call and puts) across strike prices and maturity are available daily. On the third Friday of every month, we take the put options with maturity of 28-35 days which is representative of the most liquid put option for that month. Once we fix the maturity, we pick strike prices with moneyness 1, 0.98, 0.97, and 0.95, i.e. put options with $\frac{X}{R} = 1, 0.98, 0.97, 0.95$, which comprise our cross-section. To speed-up computation we use only quarterly option prices. That is, from 1988 to 2006, we use put option prices from towards the end of the third week of January, April, July, and October. A time-series of the cross-section is presented in the top panel of Figure 5.

4.2 MCMC Methodology

We have a non-linear state space model, annual consumption data, and quarterly prices on a cross section (across strike prices) of put option prices. Our analysis is Bayesian. This enables us to input varying levels of prior information and obtain full posterior inference for both the parameters and the states.

It is possible to use both the consumption and option data in our state space model inference. However, annual consumption data is simply not very informative for parameter inference compared to the option data. In addition, they are observed at different frequencies. We opt for a simple although admittedly informal approach. We first use the option data to get posteriors for the states and all parameters except ψ and β which do not enter the option pricing equation. We then use the posteriors from the option data

[9]we look at post-1987 crash prices because the interesting mispricing anomalies vis-a-vis Black-Scholes is most prominent in the post-1987 crash period.

[10]The traded options are American, whereas we work with European prices of put options. The no-arbitrage premia for American options is $\max(1, 1/Z(\theta_t, \tau))$ where $Z(\theta_t, \tau)$ is the default-free bond price which is close to 1 for small τ. Thus, the early exercise premia for short-maturity options in our sample is minute.

to help us formulate strong priors for all parameters except ψ and β in an analysis of the consumption data. Analysis of a state-space model with just consumption growth as our observable and these strong priors on the parameters obtained from the options data then gives us posteriors for ψ and β. While strictly speaking our approach does not "coherently" use all the available information in the data, we feel it is sensible given the nature of the different data sources. We primarily rely on the relatively powerful options data and then see what the consumption data can tell us about the two utility parameters.

Posteriors are computed using Markov Chain Monte Carlo (MCMC) (see, for example, Gamerman and Lopes (2006), Geweke (2005), Johannes and Polson (2003)). We construct a Markov chain in the states θ and parameters λ such that the stationary distribution is the posterior. We run the Markov chain and use the draws to estimate desired marginals.

At the "top level" our MCMC is just a simple Gibbs sampler. Given initial values for the parameters λ we draw repeatedly from the conditionals:

$$\theta \mid \lambda \ \text{ and } \ \lambda \mid \theta$$

For example, with the options data, $\lambda_o = (\sigma, \sigma_K, \sigma_\theta, \rho, \delta, \bar{\theta}, \gamma, \mu)$, and $\theta = (\theta_1, \ldots, \theta_t, \ldots, \theta_T)$ where θ_t is the latent state for quarter t. For consumption, the parameter space $\lambda_c = (\lambda_o, \psi, \beta)$ and the states are annual.

To draw the states given the parameters $(\theta \mid \lambda)$, we use FFBS (forward-filtering, backward-sampling, Fruhwirth-Schnatter (1994), Carter and Kohn (1996)). FFBS uses a Kalman-filter like forward recursion to obtain $p(\theta_T \mid y)$ where $y = (y_1, y_2, \ldots, y_T)$. In our inference based on the option data, y_t is the four relative option prices for quarter t. For the consumption data, y_t is consumption growth for year t. Given $p(\theta_T \mid y)$, the entire state is drawn using the sequence of distributions $p(\theta_t \mid \theta_{t+1}, \theta_{t+2}, \ldots, \theta_T, y)$. The brilliant insight of Fruhwirth-Schnatter (1994) and Carter and Kohn (1996) is that because of the Markov structure, $p(\theta_t \mid \theta_{t+1}, \theta_{t+2}, , \ldots, \theta_T, y) = p(\theta_t \mid \theta_{t+1}, y)$. These conditionals can be drawn using information computed during the forward recursion when the underlying state-space is linear and normal *or* when the state is discrete. We discretize the univariate state and apply FFBS. A full description of the algorithm is in Appendix D.

We feel this simple "state-discretization" strategy is a good option which has not been exploited in the Bayesian-finance literature. While it is slow and some care is needed to organize the computations so that expensive evaluations are not done repeatedly, it is very robust. By that we mean we know that we are getting valid draws from the correct conditional without having to tune the algorithm. Of course, there is some error due to the discretization but it is minimal and easy to assess. Most of the literature (see for example

Johannes and Polson(2003)) relies on the Metropolis-Hastings (MH) algorithm. At each iteration, a proposal for a state draw is made and the proposal is stochastically accepted or rejected according to the MH recipe. The problem is that it is very difficult to know if the proposal is good enough in the case of a complex non-linear observation equation. One never knows if different results could be obtained from longer runs or different starting values. While these kinds of issues are present in virtually all MCMC implementations, we feel they are particularly problematic in our application. It may be difficult to replicate results obtained using an MH proposal for $\theta \mid \lambda$. This is not the case with our approach.

The really difficult draw is actually $\lambda \mid \theta$. Although this involves at most a ten-dimensional draw, something that seems relatively routine in the modern MCMC world, there are two fundamental difficulties. First, although we have little analytical knowledge of the option pricing equation (22), evaluation of the function using a variety of parameter values indicates a complex and non-linear relationship. This can make MCMC navigation very difficult. For a nice discussion of these issues see Geweke (2005). Secondly, a cross-section of just four options may not be informative enough about λ. Computationally, evaluation of equation (22) is by far the most time consuming step. In future work we hope to use parallel processing to dramatically speed up our MCMC by simultaneously computing prices across a larger cross-section of options. Note that the difficulty in drawing λ also makes sequential estimation via particle filtering a practically infeasible alternative. While particle filtering might work for the draw of $\theta \mid \lambda$, i.e. a pure filter case, it is not a straightforward matter to get joint inference for λ and θ, i.e a "parameter learning case". Parameter learning has received considerable attention in the recent years, but even well established parameter learning schemes, such as the ones in Liu and West (2001) and Carvalho *et. al.* (2008), would not be feasible options given the difficulty of our problem. More specifically, Liu and West's sequential normal approximation of $p(\lambda|y^t)$ is unreasonable given λ's highly nonlinear nature. Similarly, Carvalho *et al.*'s particle learning can not be implemented since the posterior distribution of λ and θ can not be represented by a small dimensional set of conditional sufficient statistics. Currently, our MCMC scheme seems to be the best available alternative.

We have again opted for a conservative strategy for drawing λ. First, we discretize so that each parameter is restricted to a grid of values. We then use Metropolis-within-Gibbs using the MH algorithm to draw either an individual parameter given the others and the states, or a group of parameters given the remaining ones and the states.

For the i^{th} element of λ, we choose a parameter s_i which we use to tune the MH proposal. Larger s_i will correspond to proposing bigger moves. We first draw each parameter one at a time and then we draw a block of parameters jointly. The trade-off here is that the "curse of dimensionality" makes proposals in larger spaces difficult to accept. On the other hand, if we only move a small subset of parameters (e.g. just one) strong dependence amongst the parameters may make movement slow (again, see Geweke (2005)).

For each draw of a single parameter we let s be one of the three values $(s_i/2, s_i, 2s_i)$ where the first and third values have probability .25 of being chosen. The idea is to randomize the size of the proposed move. Our proposal on the grid is then proportional to

$$p(\lambda_i^* \mid \lambda_i^o) \propto \exp\left(-\frac{1}{2s^2}(\lambda_i^* - \lambda_i^o)^2\right)$$

where λ_i^* denotes a proposed value for the i^{th} component and λ_i^o denotes the current value. That is, we evaluate $p(\lambda_i^* \mid \lambda_i^o)$ for each λ_i^* in our grid and then renormalize to obtain a discrete distribution on the set of grid values. Clearly, this is in the spirit of random-walk Metropolis where the proposed value is a random increment from the current value. The bigger s is, the more likely it is to draw a grid value "far" from λ_i^o. Note, however, that because we are on a finite grid, it is not a random walk and probabilities of going and coming back do not cancel out in the calculation of the MH acceptance probability.

For the joint draws, we draw either five or three parameters jointly each with probability 0.5. In the case of five, we draw $(\sigma_\theta, \rho, \delta, \bar{\theta}, \gamma)$ and in the case of three we draw $(\sigma_\theta, \delta, \bar{\theta})$. These subsets were chosen based on indications of posterior dependence from preliminary draws. For our proposal, we draw each of the parameters in the subset independently using the same approach as in the single parameter draw. We divide the corresponding s_i by three with probability 0.5 and by two with probability 0.5. We use smaller s values in the joint draw because smaller changes (on average) are needed in the individual parameters to obtain a moderate proposal in the joint space.

4.3 Empirical Results

A key advantage to the Bayesian approach is the ability to input prior beliefs. For some of our parameters we have substantial information. In section (4.3.1), we present priors and posteriors for the model parameters $(\sigma, \sigma_K, \sigma_\theta, \rho, \delta, \bar{\theta}, \gamma, \psi, \beta, \mu)$. We report results for a highly informative "tight" prior and a less restrictive "loose" prior.

In section (4.3.2) we present our inference for the states θ_t. In section (4.3.3) we present the marginal posteriors of the option prices across time and strike price and compare them to the actual values. In section (4.3.4) we discusss model implied risk premia, in (4.3.5) we discuss the effect on risk-free rate and finally in section (4.3.6) we present our fit of the annual consumption growth rates.

4.3.1 Priors and Posteriors for Model Parameters

Priors and posteriors for the model parameters are depicted in Figures 3 and 4. Posterior quantiles are given in table 1. All parameter inference is from the options data except for ψ and β which are inferred from the consumption data. The utility parameters γ, ψ, and β are shown in Figure 4, while the rest are shown in Figure 3.

For each parameter, an interval is chosen. The prior is then uniform on that interval. All parameters are independent under the prior. For all parameters except ψ and β, the interval corresponding to the loose prior is the horizontal range of the plot while the interval corresponding to the tight prior is indicated by two dashed vertical lines. For ψ and β (Figure 4) there is just one prior and the interval is the horizontal range. For example, the inference for σ_θ is in the (2,1) position of Figure 3. Under the loose prior, σ_θ is uniform on the interval (0.005,0.05). Under the tight prior, σ_θ is uniform on the interval (0.005,0.02). In each plot the marginal posterior density of the parameter obtained using the loose prior is drawn with a solid curve, and the posterior from the tight prior is drawn with a dashed curve. The inference for σ is in the (1,1) position of figure 3. Under the loose prior, the posterior is centered at 0.0023, while under the tight prior it is centered at 0.005. For ψ and β only one prior specification was used so there is a single solid curve. Note that because the posterior density estimates are kernel smooths of the MCMC draws, they appear to extend beyond the prior support.

The tight prior is chosen to make the underlying growth rates highly autocorrelated. Indeed, in the long-run risk literature starting from Bansal and Yaron (2004), the assumption has been that exposure to strongly autocorrelated states gives rise to substantial long-run risk. Highly autocorrelated states corresponds to low values of δ and σ_θ, and we impose that through our priors as can be seen by the dashed vertical lines in the subplots in the (2,1) and (3,1) positions of Figure 3. At the same time, we also impose a prior to restrict the size of the transient shock. This corresponds to smaller values of σ_K. The loose prior setting lifts these restrictions and lets the data dictate where the parameters lie. Lifting these restrictions, especially for the autocorrelation of the growth rates, allows us to generate high time-varying risk-premia. This is because the time-variation in our risk-premia is greatly magnified for low levels of autocorrelation since the loading on the stochastic component of the risk-premia, $\tilde{c} \propto \delta$. Our tight prior on γ restricts it to be less than 9 as opposed to 15 under the loose prior.

The parameter inference is markedly different under the loose prior with only the marginal posterior for ρ being similar under the two specifications. The posterior of σ indicates that we have much smaller pricing error under the loose prior. The posteriors for δ, σ_θ, and σ_K indicate that we obtain this improved fit by modeling the state θ_t as less autocorrelated and more noisy than is suggested by the tight prior. The marginal prior for ψ is also consistent with $\psi < 1$. Recall that with $\psi < 1$ our solution to the value function is exponentially quadratic in θ_t. We will see in the next two sections that the relatively noisy specification for the state process under the loose prior, coupled with the implications of our exponentially quadratic solution to the value function, allows us to fit the option prices well.

4.3.2 Put Option implied growth rates

A strength of our Bayesian filtering mechanism is that we can take the panel of observed relative put option prices and filter from the system a full distribution of growth rates at each point in the time-series. Thus, we can get a full posterior distribution of the underlying latent state θ_t for each time t.

Figure 5 plots our options data in the top panel, the filtered states from the loose prior in the middle panel, and the states from the tight prior in the bottom panel. In the bottom two panels, the dot is plotted at the posterior mean and the vertical dashed line through each dot indicates a 95% marginal posterior interval.

The states precisely follow the pattern that the model predicts. High prices of put option insurance reveals strong precautionary savings motivation in bad states. Thus, the corresponding economic states are low. Precisely the opposite takes place under high growth rates when precautionary savings motivation dissipates. The price of put option is low, and the underlying economic growth rate is high. In the next section, we show that this effect is brought about by time-variation in the "risk-adjusted" probabilities that a put option finishes in the money. In bad states, the probability is higher and thus the put option commands a high price. The time-series of the states reflect the priors that we impose in our econometric exercise. The states from the tight prior are much smoother reflecting the strong prior about small values for δ and σ_θ.

Many authors have used jumps and/or stochastic volatility to build option pricing models. The middle panel of Figure 5 indicates that while jumps in our state θ_t are possible, the simple specification used in this paper is not unreasonable. While states appear to be noisy around 2000, we clearly cannot reject the hypothesis that our simple specification for θ_t is reasonable. We will see in the next section that our general equilibrium approach enables us to fit the put option prices with this relatively simple model of the underlying state evolution.

4.3.3 Time-Series and Cross-Sectional Fits of Option Prices

The overall fit under the two prior settings is shown in Figure 6. The posterior means of the option prices are plotted against the data. The results from the loose prior are in the left panel and the results from the tight prior are in the right panel. The four different symbols indicate the four different moneyness we consider in our sample. The loose prior does significantly better in lining up against the 45-degree line than the tight prior indicating superior fit. The closeness of fit by the loose prior is further shown by the pricing error σ in the observation equation of option prices, whose 95% posterior confidence band is between $(0.002, 0.003)$. In the case of the tight prior, the confidence band for the pricing error is much higher at $(0.004, 0.006)$ (see Figure 3).

An important feature of the post-1987 market crash is the volatility smirk which is an implication of the failure of the Black-Scholes model to price OTM put options. That problem manifests itself through the well-known put option smirk, which is essentially a one-to-one mapping from higher prices to higher volatility of the underlying security as implied by the risk-neutral dynamics of the put options. In the post-1987 period the implied volatility from OTM put options shows a noticeable smirk, which suggests that the underlying volatility (or risk-premia) extracted from OTM put options is much higher than Black-Scholes, which uses constant volatility throughout the cross-section. The entire smirk may move up or down - a time-series phenomenon - but the general shape of the smirk is a failure of Black-Scholes to generate high prices for OTM put options.

We are able to address this phenomenon and improve upon Black-Scholes in this framework. Figure 7 graphically shows that we are able to match the cross-sectional average prices a lot closer than Black-Scholes under both loose and tight prior settings.[11] The values for the loose and tight priors are obtained by simply averaging over time. Since we are able to match put prices, we are also able to match the smirk since their relationship is one-to-one. In this regard we are similar to the stochastic volatility model of Heston (1993) which can also generate high OTM put option prices under certain parametrization. The big distinction from Heston (1993) is that we endogenize the time-variation in our risk-premia. The stochastic volatility in our marginal utility process, which reflects the strong precautionary savings motivation in bad states and high satiation and low risk prices in good states, is determined within our general equilibrium setting. This generates the corresponding state-dependence in option prices, just like the Heston (1993) model does with exogenous specification of stochastic volatility.

[11]The Black-Scholes prices are computed from time-series averages using the average annualized nominal interest rate on the 30-day T-bill which is 4%, and using volatility of 0.15, which is the average annualized volatility of S&P index in this time-period.

The time-series fit with the panel of put options is shown in Figure 8. The posterior distribution of option prices (the heavy bands) is plotted against the actually observed option price (the dot) at every date on the sample. The tight prior is marked with a dark band while the loose prior is the lighter band, where the bands represent the 95% posterior intervals. The loose prior setting does substantially better in fitting the OTM put options. The tight prior with extremely autocorrelated states provides particularly poor fit in bad economic states when put option prices are high. The loose setting with less autocorrelated states is able to match the high put option prices in bad times substantially better. Overall, the tight prior setting with very autocorrelated states simply cannot generate high enough premia in bad times to be able to match the high put option prices. It can still do relatively well in matching the average put option prices, but clearly fails in generating the strong time-series patterns under the loose setting.

The key question is how is it that the loose prior setting able to capture such strong time-series as well as cross-sectional effects? It is because the loose prior setting is able to generate high probabilities for the put option to finish in the money across all levels of moneyness and economic state than the tight prior as is shown in Table 3. First of all, notice that for bonds with maturity of one month in this setting, the price is very close to one across all reasonable states. Thus, it is not the risk-free rate effect that generates the time-series phenonmenon in put option prices. The answer lies in the non-linear relationship between the current state θ_t and the moneyness X/R which is summarized through the risk-adjusted probability $1 - P2(g_t, \theta_t, \tau)$ (where $g_t = \ln(R_t)$) that the put option finishes in the money. The top panel of Table 3 takes the median parameter estimates from the posterior distribution of the tight prior setting and computes the risk-adjusted probability that the put option finishes in the money - $1 - P2$, across different states. The bottom panel of Table 3 does the same with the loose prior setting. At each level of moneyness, the probability that this particular put option finishes in the money is higher for the loose prior setting than the tight one across all economic states. For example, take $X/R = .98$, or the 2% OTM put option. The probability that this put option finishes in the money is 29.4% in a bad economic state ($\theta_t = -0.03$) and 24.3% in a good economic state ($\theta_t = 0.03$) using the parameters of the loose prior setting. Using the parameters of the tight prior setting, the corresponding probabilities for the 2% OTM put option is 9.1% at $\theta_t = -0.03$ and 4.9% at $\theta_t = 0.03$. The effect is more severe for more OTM put options. Three things are clear from Table 3. First, the time-series effect of put option prices can be replicated using the stochastic growth rates - low (high) growth rates imply a higher (lower) probability that the put option will finish in the money, and hence the counter-cyclical pattern of prices. Secondly, the loose prior setting can generate much higher probabilities for the put options to finish in the money thus ensuring high put option prices, especially in bad states, that matches the empirical evidence. Finally, the joint evidence with Figure 7, is that the non-linear relationship between the growth rate θ_t and moneyness captured in the risk-adjusted probability $1 - P2$ can be used to capture the volatility smirk.

4.3.4 Stochastic Risk-Premia

The reason the loose prior setting can generate substantial risk-adjusted probability of put options finishing in the money is because it can generate substantial risk-premia from precautionary savings. In fact, (17) shows that due to the exponentially quadratic solution of the value function, the risk-premia should be stochastic and counter-cyclical. Taking the full posterior distribution of the parameters and the time-series of the underlying growth rates θ_t, we construct the full distribution of the long-run risk-premia according to (17) as is inferred by put option prices. The top panel of Figure 9 plots the long-run risk-premia corresponding to the loose setting, while the bottom panel plots the same for the tight setting. The long-run risk-premia is substantial in both settings. In the tight setting, it is substantially smoother, but that is entirely due to the fact that the underlying states are smooth. The loose setting generates much less autocorrelated risk-premia, and its time-variation is substantial. In fact, it can reach as high as 30% in times of low growth rates and as low as -10% in good states. In comparison, from the smooth growth rates of the tight setting, risk-premia stays between 10-20% across good and bad states. The reason for such high time-variation for the loose setting is obvious. The loading on the stochastic component for long-run risk premia is $\tilde{c} = \frac{2\delta + h_1}{\sigma_\theta^2}$, where δ is the autocorrelation parameter and smaller δ implies higher autocorrelation. As such, higher the δ, lower the autocorrelation and higher the risk-premia. We can generate high long-run risk-premia from low levels of autocorrelation, which is contrary to the current literature like Drechsler and Yaron (2007) and Shaliastovich (2008). In Drechsler and Yaron (2007), the growth rate of consumption is extremely autocorrelated which helps to generate substantial premia. Shaliastovich (2008) has short-run dynamics but its contribution to overall premia is small compared to his other channels.

Table 2 shows the breakdown of the long-run risk-premia implied by put options into the constant component and the stochastic component. First of all, the transient component of risk due to iid production shocks is different across two prior settings reflecting the underlying prior assumptions. The tight setting not only restricted δ to be small, but also restricted the size of σ_K - the volatility of the transient shock. Thus, the posterior median of the transient risk-premia is small at 0.6% in the tight prior case. In the loose prior setting, the transient shock is allowed to be higher. The posterior median of σ_K is 0.09

which matches the calibrated value of σ_K in Ai (2009). It produces a median transient risk-premia of 5.8%. If that is considered a substantial risk-premia from transient risk, consider the long-run risk contribution in each of the prior setting. At low growth rates, the long-run risk premia can be as high as 30% in the loose setting with less autocorrelated states, and 20% in more autocorrelated states. Thus, even if we generate high transient risk the long-run risk component is even bigger. The loading on the stochastic component is substantial - 5.75 in the loose prior setting and 2.77 in the tight prior setting. That implies that if growth rates decrease by one unit, risk-premia shoots up by 5.75% in the loose prior setting and 2.77% in the tight setting. Clearly the stochastic component of the long-run risk-premia implied by put option prices is considerable in the economy.

The risk-premia in Figure 9 is also clearly counter-cyclical. The corresponding economic growth rates are given in Figure 5. Higher growth rates imply lower risk-premia and vice-versa. In bad times, the high precautionary savings motivation is consistent with high risk-premia. The agent feels more risk-averse at a time when the probability of a wealth loss is high. Clearly, the agent's demand for insurance is high and he is willing to pay more for put options in these states. Likewise, in good states the precautionary savings motive dissipates and the agent's risk-premia is low. In these states, the agent is not fearful of a wealth loss and his willingness to pay for put options to insure his wealth is low. This explains the counter-cyclical risk-premia pattern that we filter out of the time-series of put option prices.

4.3.5 Risk-free rate

The time-variation in risk-premia also produces a corresponding time-variation in the risk-free rate whose volatility is, unfortunately, quite high. Based on the expression of the risk-free rate in (16), the unconditional average and volatility of the risk-free rate is given by

$$\begin{aligned} E(r_t^f) &= \frac{(\gamma-1)\sigma_K\sigma_\theta\rho(1+\tilde{c}\sigma_K\sigma_\theta\rho)}{\delta} + \bar{\theta} - \gamma\sigma_K^2 \\ Vol(r_t^f) &= (1+\tilde{c}\sigma_K\sigma_\theta\rho)\frac{\sigma_\theta}{\sqrt{2\delta}} \end{aligned}$$

The $\tilde{c}$ term that was responsible for time-varying risk-premia also introduces high volatility in the risk-free rate. It illustrates the fluctuation in the agent's preference for certainty equivalence from low to high growth rate states. Clearly, if we proceed with the exponentially linear solution of the value function, then $\tilde{c}$ would have been zero and that would have significantly reduced the mean and volatility of the risk-free rate. Taking the marginal posterior distribution of the parameters under the two prior settings, we compute the posterior mean and volatility of the risk-free rate under each setting. Under the loose setting, the posterior mean of the risk-free rate is between 2.5-13.6%, whereas under the tight setting, the posterior mean is between 3.3-6.3%. Under the loose setting, the posterior volatility of the risk-free rate is between 10.6-12.5%, whereas under the tight setting the volatility is between 4.3-4.8%. Clearly, the loose prior setting that generates large fluctuations in risk-premia from less autocorrelated states bring forth considerable volatility of the risk-free rate. The tight setting which produces smoother states and less time variation in risk-premia obviously generates less volatility in the risk-free rate as well. There is a clear statistical reason why the loose prior setting generates such high mean and volatility of the risk-free rate. The loose setting generates stronger fit with option prices for which the stochastic risk-premia component needs to be accentuated. The tight setting does not seek to fit the option data too strongly resulting in far less emphasis on the time-variation component of the risk-premia. Table 2 shows that the median value of $\tilde{c}\sigma_K\sigma_\theta\rho$ - the term responsible for time-variation in risk-premia is 5.75 under the loose prior and 2.77 under the tight prior.

However, the theory itself predicts that the risk-free rate ought to be quite volatile. As Weil (1990) points out, early resolution of uncertainty is accompanied with certainty equivalent fluctuations of utility over time that are of large amplitude. The substitution effect under good times depresses the marginal propensity to save thereby increasing the risk-free rate. The income effect in bad times is simply a precautionary savings effect that increases the propensity to save thus lowering the risk-free rate. Option prices reveal large time-variation in risk-premia, and by extension, in the price of certainty equivalence. This generates a large volatility of the risk-free rate which we infer by fitting the option prices very closely. As has been pointed out earlier, the high volatility of risk-free rate is also observed in other preferences that explore non-separability, like habit formation a la Santos and Veronesi (2008). In fact, Campbell and Cochrane (1999) imposes parameter restrictions in their habit formation model to shut down volatility of the risk-free rate.

4.3.6 Consumption Growth

The time-series pattern of optimal consumption is given by (13). It is derived endogenously from the first order condition of the agent under SDU and fluctuating economic growth. As mentioned above, the data of aggregate consumption simply isn't informative for the structural parameters in our model, so we use the rich option data to infer the parameter estimates of our model. However, there are two parameters - ψ and β, that cannot be inferred from option prices and we have to get their estimate them from aggregate consumption.

First, the only prior setting we consider here is the loose prior setting which fits the options data well. The way we proceed with the loose prior setting is that we impose a very tight prior restriction on the parameters that can be estimated from option prices. In other words, we take the posterior distribution of parameters estimated from option prices under the loose prior setting and use them as strong priors for analyzing consumption growth. This informal approach gives our estimation a general equilibrium flavor in the sense that we are trying to impose discipline in the parameter set to be restricted to the parameter space that determined option prices.

Under the above prior setting on λ_o, we estimate the posterior distribution of ψ and β from consumption growth and report them in Table 1. It shows that the posterior distribution of ψ has a median of 0.90 with a 95% posterior interval of $(0.623, 0.949)$ and β has a median estimate of 0.02 with posterior interval between $(0.00, 0.04)$. As Figure 4 indicates, the posterior distribution of ψ peaks strongly at 0.9 showing that the consumption data is quite informative about ψ, whereas the β parameter is relatively uninformative with a very wide distribution. More importantly, we conclude what we have previously hypothesized - that the Intertemporal Elasticity of Substitution of the agent is less than one which was necessary for the quadratic solution of the value function to go through that gave rise to substitution effect, time-varying risk-premia, etc. There is quite a bit of controversy about the estimation of ψ in the literature. Our inference is closer to Hall (1988), Campbell (1999) and Vissing-Jorgensen (2002) who find that $\psi < 1$. On the other hand, Attanasio and Weber (1989) conclude that $\psi > 1$, whereas Yogo (2004) estimates ψ in several countries and concludes that one cannot reject unit IES in every country except Canada and Switzerland. In Bansal and Yaron (2004), $\psi > 1$ generates high risk-premia and our conclusion is different from theirs.

As a by-product of the estimation of ψ and β, we also filter the expected growth rates of consumption $\mu_C(\theta_t)$ in (13) which is a highly non-linear function of θ_t and compare it against actual consumption growth in Figure 10. Clearly, the same set of parameters that fit the options data very well produce expected consumption growth that reasonably matches the observed time-series. The solid line through the center of the data is the expected consumption growth rate $\mu_C(\theta_t)$ with the dashed line showing the inter-quartile range of expected consumption growth.

Taking the posterior distribution of the parameters under the loose prior setting and the time-series of θ from the option prices, we compute the posterior distribution of the consumption to capital ratio given in proposition 1. Its posterior distribution is shown in Figure 11. Since we are using the loose prior setting where the states are far less autocorrelated, our consumption to capital ratio has plenty of time-series variation. In the low growth rate/high risk-premia states, an agent with preference for early resolution of uncertainty has a strong preference for certainty equivalence. Thus, the risk-free rate is low, but the agent still saves more out of precautionary savings motive. In low growth states, an agent facing stochastic growth rates is exposed to the possibility of even worse times in the future and he guards against that by saving more relative to consumption. Higher savings relative to consumption lowers the consumption to capital ratio, and thus we observe the ratio dropping as a response to signal of poor economic growth. In high growth rate states, the precautionary savings motive disappears and the substitution effect prevails over the income effect. In high growth states, the agent is forecasting high consumption growth in the future. To keep utility stable, an agent with low elasticity of intertemporal substitution increases consumption relative to savings which increases the consumption to capital ratio. To reiterate, the above phenomenon is strictly due to the exponentially quadratic solution of the value function. The exponentially linear solution makes the consumption to capital ratio monotonic in the underlying state. In this case, for an agent with $\gamma > 1$ and $\psi > 1$ ($\psi < 1$), the substitution (income) effect always prevails over the income (substitution) effect. In the quadratic case, which effect will dominate also depends on the underlying state, and as we show we can bring about substitution effect with $\psi < 1$ in good economic states whereas the income effect dominates the substitution effect in poor states.

This completes our goal. Our hypothesis that the counter-cyclical time-series pattern of option prices can be rationalized through SDU, intertemporal elasticity of substitution less than one, and time-varying production shock giving rise to counter-cyclical premia is hereby confirmed. Furthermore, our theory is also consistent with observed data on consumption growth.

5 Conclusion

A general equilibrium model of insurance motivation is presented under stochastic differential utility and time-varying production growth rates. They produce counter-cyclical risk-premia large enough to explain the time-variation in prices of put options - our proxy for demand of insurance. The essential feature of the model needed to bring out this effect is state-dependent income or substitution effect that generates strong precautionary savings motive and high risk-premia in bad times, and high degree of satiation and low risk-premia in good times. The state-dependence of risk-premia is solely due to low values of intertemporal elasticity of substitution, and thus we fully realize the benefit of separating risk-aversion from intertemporal substitution which stochastic differential utility allows us to do. Moreover, we generate time-varying risk-premia from simple dynamics of underlying economic growth without introducing stochastic volatility or jumps into our underlying driving processes. We estimate our structural model and infer that we can explain the time-variation in put option prices with relatively low risk-aversion and time-varying long-run risk. The latter is realized through low levels of autocorrelation in the growth rates compared to the rest of the long-run risk literature implying that the power of our model is derived out of short-run dynamics. Using a Bayesian computational scheme, we employ a non-linear filtering mechanism to obtain growth rates out of put option prices and confirm that these contracts reveal counter-cyclical premia.

This paper takes a full general equilibrium model straight to the data preserving all the non-linearities induced by the structural setting. The theory along with the estimation gives us deep insight about the risk-preferences in the underlying economy and dynamics that can explain key asset-pricing questions. The computational approach is rather flexible and can be applied to a whole host of equilibrium problems in economics and finance.

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6 Appendix

6.1 A: Equilibrium

Solution of planner's problem

The HJB equation satisfying the social planner's problem is

$$0 = \max_C f(C,J) + J_\theta \delta(\bar{\theta} - \theta) + J_K(K\theta - C) + \frac{1}{2}J_{\theta\theta}\sigma_\theta^2 + \frac{1}{2}J_{KK}K^2\sigma_K^2 + J_{K\theta}K\sigma_K\sigma_\theta\rho \quad (24)$$

The solution is

$$C^* = J_K^{-\psi}\beta^\psi \left[(1-\gamma)J\right]^{\frac{1-\psi\gamma}{1-\gamma}} \quad (25)$$

Plugging it back produces

$$\begin{aligned} \frac{(1-\gamma)J}{\psi - 1}\left[\frac{\beta^\psi}{J_K^{\psi-1}}\left[(1-\gamma)J\right]^{\frac{\gamma(1-\psi)}{1-\gamma}} - \psi\beta\right] + J_\theta\delta(\bar{\theta}-\theta) + \\ J_K K\theta + \frac{1}{2}J_{\theta\theta}\sigma_\theta^2 + \frac{1}{2}J_{KK}K^2\sigma_K^2 + J_{K\theta}K\sigma_K\sigma_\theta\rho = 0 \end{aligned} \quad (26)$$

Notice that the value function J is homogeneous of degree $1-\gamma$ in K, and thus propose a solution $J = \frac{H(\theta)K^{1-\gamma}}{1-\gamma}$. That reduces the above PDE to an ODE

$$\begin{aligned} \frac{\beta}{\psi-1}\left[\beta^{\psi-1}H(\theta)^{\frac{1-\psi}{1-\gamma}} - \psi\right] + \theta + \left[\frac{\delta(\bar{\theta}-\theta)}{1-\gamma} + \sigma_K\sigma_\theta\rho\right]\frac{H'(\theta)}{H(\theta)} + \\ \frac{1}{2}\frac{\sigma_\theta^2}{1-\gamma}\frac{H''(\theta)}{H(\theta)} - \frac{1}{2}\gamma\sigma_K^2 = 0 \end{aligned} \quad (27)$$

which means $C^* = y^* = \beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}}K$.

An approximate solution is available by linearizing the leading term of (27) around the long-run consumption to capital ratio for $\psi \neq 1$. Let $H(\theta)^{\frac{1-\psi}{1-\gamma}} = I(\theta)^{-1}$. Notice that the optimal consumption condition implies

$$\begin{aligned} \frac{C^*}{K^*} &= e^{c-k} \\ &\approx e^\mu(1-\mu) + e^\mu(c-k) \\ \beta^\psi I(\theta)^{-1} &= e^\mu(1-\mu) + e^\mu\left(\psi\ln\beta - \ln I(\theta)\right) \end{aligned}$$

where $c = \ln C$, $k = \ln K$, $\mu = E(c-k)$. Substituting that in (27) simplifies the above expression to

$$\frac{1}{\psi-1}\left[h_0+h_1(\psi\ln\beta-\ln I(\theta))-\beta\psi\right]+\theta+\left[\frac{\delta(\bar{\theta}-\theta)}{1-\gamma}+\sigma_K\sigma_\theta\rho\right]\frac{1-\gamma}{\psi-1}\frac{I'(\theta)}{I(\theta)}$$
$$+\frac{\sigma_\theta^2}{2}\left[\frac{1}{\psi-1}\frac{I''(\theta)}{I(\theta)}-\frac{\gamma+\psi-2}{(\psi-1)^2}\frac{I'(\theta)^2}{I(\theta)^2}\right]-\frac{1}{2}\gamma\sigma_K^2=0$$

where $h_0=h_1(1-\ln h_1)$ and $h_1=\exp\mu$. The solution to the above is $I(\theta)=\exp\left(a+b\theta+\frac{1}{2}c\theta^2\right)$. Substituting that in the above and collecting terms, leaves three equations in three unknowns $\{a,b,c\}$ which can be solved sequentially.

$$\begin{aligned}
c &= \frac{\psi-1}{1-\gamma}\frac{h_1+2\delta}{\sigma_\theta^2} \qquad \text{OR} \qquad 0\\
b &= \frac{(\psi-1)((\psi-1)+c(\delta\bar{\theta}+(1-\gamma)\sigma_K\sigma_\theta\rho))}{(h_1+\delta)(\psi-1)-c\sigma_\theta^2(1-\gamma)}\\
a &= \frac{1}{h_1}\left[-\frac{(\psi-1)\gamma\sigma_K^2}{2}+\frac{1}{2}\sigma_\theta^2\left(\frac{1-\gamma}{\psi-1}b^2+c\right)+(\delta\bar{\theta}+(1-\gamma)\sigma_K\sigma_\theta\rho)b+h_1\psi\ln\beta+h_0-\beta\psi\right]
\end{aligned}$$

Thus, the value function becomes $J(K_t,\theta_t)=\frac{K_t^{1-\gamma}}{1-\gamma}H(\theta_t)=\frac{K_t^{1-\gamma}}{1-\gamma}I(\theta_t)^{\frac{1-\gamma}{\psi-1}}=\frac{K_t^{1-\gamma}}{1-\gamma}e^{\left(\tilde{a}+\tilde{b}\theta_t+\frac{1}{2}\tilde{c}\theta_t^2\right)}$ where

$$\begin{aligned}
\tilde{c} &= \frac{h_1+2\delta}{\sigma_\theta^2} \qquad \text{OR} \qquad 0\\
\tilde{b} &= \frac{(1-\gamma)+\tilde{c}(\delta\bar{\theta}+(1-\gamma)\sigma_K\sigma_\theta\rho)}{h_1+\delta-\tilde{c}\sigma_\theta^2}\\
\tilde{a} &= \frac{1}{h_1}\left[\frac{1}{2}\sigma_\theta^2(\tilde{b}^2+\tilde{c})+(\delta\bar{\theta}+(1-\gamma)\sigma_K\sigma_\theta\rho)\tilde{b}+\frac{1-\gamma}{\psi-1}\left[h_1\psi\ln\beta+h_0-\beta\psi-\frac{(\psi-1)\gamma\sigma_K^2}{2}\right]\right]
\end{aligned}$$

As is shown in Figure 2, the non-zero root above is a better solution of (27) for $\psi<1$, but the zero-root starts is a better solution for $\psi>1$. For $\psi>1$, the zero root implies that

$$\begin{aligned}
\tilde{c} &= 0\\
\tilde{b} &= \frac{(1-\gamma)}{h_1+\delta}\\
\tilde{a} &= \frac{1}{h_1}\left[\frac{1}{2}\sigma_\theta^2\tilde{b}^2+(\delta\bar{\theta}+(1-\gamma)\sigma_K\sigma_\theta\rho)\tilde{b}+\frac{1-\gamma}{\psi-1}\left[h_1\psi\ln\beta+h_0-\beta\psi-\frac{(\psi-1)\gamma\sigma_K^2}{2}\right]\right]
\end{aligned}$$

This is the solution corresponding to the full-information economy of Ai (2007) who does not consider the quadratic solution of the value function. This is nested in our setting for the special case where $\tilde{c}=0$.

We can recover the well-known closed-form solution for $\psi=1$. The zero-root starts to be a better solution of the ODE in (27) as $\psi \to 1$. But as $\psi \to 1$, $\mu \to \ln\beta$ according to Propostion 1 (since the consumption to capital ratio is a constant β in the $\psi = 1$ case), and

$$\begin{aligned}
\lim_{\psi\to1} \tilde{c} &= 0 \\
\lim_{\psi\to1} \tilde{b} &= \frac{1-\gamma}{\beta+\delta} \\
\lim_{\psi\to1} \tilde{a} &= \frac{1}{\beta}\left[\frac{1}{2}\sigma_\theta^2\tilde{b}^2 + (\delta\bar{\theta} + (1-\gamma)\sigma_K\sigma_\theta\rho)\tilde{b} + (1-\gamma)\left[\beta\ln\beta - \beta - \frac{\gamma\sigma_K^2}{2}\right]\right]
\end{aligned}$$

Furthermore, if $\gamma \to 1$ and $\psi \to 1$, both $\tilde{a} = \tilde{b} = 0$, and we recover the log utility case.

6.2 B: Asset Pricing

The Pricing Kernel

Duffie and Epstein (1992) has shown that the state-price density for recursive preferences is given by

$$\Lambda_t = \exp\left[\int_o^t f_J(C_s, J_s)ds\right] f_c(C_t, J_t) \tag{28}$$

For the value function $J = \frac{H(\theta)K^{1-\gamma}}{1-\gamma}$,

$$\begin{aligned}
f_C &= K^{-\gamma}H(\theta) && (29)\\
f_J &= \frac{\beta\psi}{\psi-1}\left[\beta^{\psi-1}H(\theta)^{\frac{1-\psi}{1-\gamma}}\left(\frac{1}{\psi}-\gamma\right) - (1-\gamma)\right] && (30)
\end{aligned}$$

So,

$$\begin{aligned}
\frac{d\Lambda}{\Lambda} &= \frac{df_C}{f_C} + f_J dt && (31)\\
&= -r_t^f dt - \gamma\sigma_K dB_{Kt} + \frac{H'(\theta)}{H(\theta)}\sigma_\theta dB_{\theta t} && (32)
\end{aligned}$$

where r_t^f is the risk-free rate and is equal to

$$\begin{aligned}
r_t^f = &-\frac{\gamma(\gamma+1)}{2}\sigma_K^2 - \frac{\psi}{\psi-1}\left[\beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}}\left(\frac{1}{\psi}-\gamma\right) - \beta(1-\gamma)\right] + \\
&\gamma\left[\theta - \beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}}\right] - \frac{H'(\theta)}{H(\theta)}\left(\delta(\bar{\theta}-\theta) - \gamma\sigma_K\sigma_\theta\rho\right) - \frac{\sigma_\theta^2}{2}\frac{H''(\theta)}{H(\theta)}
\end{aligned} \tag{33}$$

and the equilibrium risk premia is given by

$$\begin{aligned} -Cov\left(\frac{d\Lambda}{\Lambda}, \frac{dP}{P}\right) &= -Cov\left(\frac{d\Lambda}{\Lambda}, \frac{dK^*}{K^*}\right) \\ &= \gamma\sigma_K^2 - \frac{H'(\theta)}{H(\theta)}\sigma_\theta\sigma_K\rho \end{aligned}$$

Therefore, the pricing kernel, risk-premia and the risk-free rate can be obtained by plugging in the specific H function obtained in the approximation.

$$\begin{aligned} \frac{d\Lambda}{\Lambda} &= r_t^f dt - \gamma\sigma_K dB_K + (\tilde{b} + \tilde{c}\theta_t)\sigma_\theta dB_\theta \\ \tilde{r}_t^P &= \gamma\sigma_K^2 - (\tilde{b} + \tilde{c}\theta_t)\sigma_K\sigma_\theta\rho \\ r_t^f &= \left[\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2\right] + (1 + \tilde{c}\sigma_K\sigma_\theta\rho)\theta_t \end{aligned}$$

By substituting in $\tilde{b}$ and $\tilde{c}$ for the $\psi < 1$ case, they can be written as

$$\frac{d\Lambda}{\Lambda} = r_t^f dt - \gamma\sigma_K dB_K - \left(\frac{1-\gamma}{\delta}(1 + \tilde{c}\sigma_K\sigma_\theta\rho) + \tilde{c}(\bar{\theta} - \theta_t)\right)\sigma_\theta dB_\theta \tag{34}$$

$$\tilde{r}_t^P = \gamma\sigma_K^2 + \left(\frac{1-\gamma}{\delta}(1 + \tilde{c}\sigma_K\sigma_\theta\rho) + \tilde{c}(\bar{\theta} - \theta_t)\right)\sigma_K\sigma_\theta\rho \tag{35}$$

$$r_t^f = \left(\frac{\gamma-1}{\delta}(1 + \tilde{c}\sigma_K\sigma_\theta\rho)\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2\right) + \theta_t + \tilde{c}\sigma_K\sigma_\theta\rho(\theta_t - \bar{\theta}) \tag{36}$$

6.3 C: Equilibrium Options pricing

Let's start with the stochastic processes satisfying aggregate wealth on the capital stock and the underlying growth rate θ_t. (The *'s are dropped from the optimal quantities for readibility).

$$\begin{aligned} \frac{dR}{R} &= \theta_t dt + \sigma_K dB_K \\ d\theta_t &= \delta\left(\bar{\theta} - \theta_t\right)dt + \sigma_\theta dB_\theta \end{aligned}$$

Any derivative $G(R_t, \theta_t, \tau)$ that matures at $\tau = T - t$ satisfies the equilibrium condition

$$E_t[dG] = r_t^f G - GE_t\left[\frac{d\Lambda}{\Lambda}\frac{dG}{G}\right] \tag{37}$$

The first term on the right is the traditional risk-neutrality term which implies that the asset G grows at the risk-free rate. The second term is the adjustment for risks - one from the traditional transient component and the other from the long-run component. For an European style call option, the boundary condition must satisfy $G(R_T, \theta_T, \tau = 0) = \max[R_T - X, 0]$ if X is the strike price of the put option.

Define a variable $g_t = \ln(R_t)$. So, the process for g_t is

$$dg = \left[\theta_t - \frac{1}{2}\sigma_K^2\right] dt + \sigma_K dB_K$$

Applying Ito's Lemma to (37),

$$\begin{aligned} G_R R\theta + G_\theta \delta\left(\bar{\theta} - \theta\right) + \frac{1}{2}\left[G_{RR}R^2\sigma_K^2 + G_{\theta\theta}\sigma_\theta^2\right] + G_{R\theta}R\sigma_K\sigma_\theta\rho - G_\tau = \\ Gr_t^f - G_R R\left((\tilde{b} + \tilde{c}\theta)\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2\right) - G_\theta\left((\tilde{b} + \tilde{c}\theta)\sigma_\theta^2 - \gamma\sigma_K\sigma_\theta\rho\right) \end{aligned}$$

Notice that in the PDE for traditional option pricing, the expected return on stocks - time-varying or not - does not appear, because under the risk-neutral measure the stock simply grows at the risk-free rate. In this case, the endogenously determined market prices of risk provide the adjustment for risk-neutrality. Collecting terms, one could write the above in the traditional risk-neutral way by accounting for the market price of risk.

$$\begin{aligned} G_R R\left(\theta - (\gamma\sigma_K^2 - (\tilde{b} + \tilde{c}\theta)\sigma_K\sigma_\theta\rho)\right) + G_\theta\left(\delta(\bar{\theta} - \theta) - \left(\gamma\sigma_K\sigma_\theta\rho - (\tilde{b} + \tilde{c}\theta)\sigma_\theta^2\right)\right) + \\ \frac{1}{2}\left[G_{RR}R^2\sigma_K^2 + G_{\theta\theta}\sigma_\theta^2\right] + G_{R\theta}R\sigma_K\sigma_\theta\rho - G_\tau = Gr_t^f \end{aligned} \tag{38}$$

The first term on the left hand side of (38) is the traditional option pricing term where the expression inside the paranthesis is exactly the risk-free rate in (36). The second term provides the risk-adjustment for the growth rate. Guess a solution similar to Black-Scholes' model of the form

$$G(R, \theta, t) = RP1(g, \theta, \tau) - XP2(g, \theta, \tau) \tag{39}$$

where $\tau = T - t$ is the time to maturity of the option. Substituting the solution (39) into the PDE, (38) reduces to

$$RA_t \cdot P1(g, \theta, \tau) - XA_t \cdot P2(g, \theta, \tau) = 0 \tag{40}$$

where A_t is a second order differential operator. This relationship will hold if the resulting diffferential equations are both equal to zero. The functional form of the PDE for $P1$ is

$$\begin{aligned} P1_g\left(r_t^f + \frac{\sigma_K^2}{2}\right) + P1_\theta\left(\delta(\bar{\theta} - \theta) + (1 - \gamma)\sigma_K\sigma_\theta\rho + (\tilde{b} + \tilde{c}\theta)\sigma_\theta^2\right) \\ + P1_{gg}\frac{\sigma_K^2}{2} + P1_{\theta\theta}\frac{\sigma_\theta^2}{2} + P1_{g\theta}\sigma_K\sigma_\theta\rho - P1_\tau = 0 \end{aligned} \tag{41}$$

and for $P2$ is

$$\begin{aligned} P2_g\left(r_t^f - \frac{\sigma_K^2}{2}\right) + P2_\theta\left(\delta(\bar{\theta}-\theta) - \gamma\sigma_K\sigma_\theta\rho + (\tilde{b}+\tilde{c}\theta)\sigma_\theta^2\right) \\ +P2_{gg}\frac{\sigma_K^2}{2} + P2_{\theta\theta}\frac{\sigma_\theta^2}{2} + P2_{g\theta}\sigma_K\sigma_\theta\rho - P2_\tau - r_t^f P2 = 0 \end{aligned} \tag{42}$$

The existence of the above map depends on the continuity and boundedness of the function which will be assumed through the standard restrictions of the underlying markov process. The uniqueness of the solution depends on the convergence to the appropriate boundary condition which is now addressed.

At expiry, i.e. at $\tau = 0$, the boundary condition for $P1$ is $P1(g_T, \theta_T, 0) = 1_{\{g_T \geq \ln(X_T)\}}$. (The same logic applies for $P2$ and is omitted). It can be shown that $P1$ is the conditional probability that the option expires in the money, i.e.

$$P1(g, \theta, 0) = \Pr(g_T \geq \ln(X)|g_t, \theta_t)$$

This probability is not available in a tractable form due to the discontinuity of the boundary condition. However, the characteristic function of the probability distribution of g_T can be obtained in closed-form and that can be inverted to obtain the probability density of g_T. Assume that another function $f(g, \theta, \tau)$ satisfies (41) with the boundary condition $f(g_T, \theta_T, 0) = \exp(i\phi g_T)$, which is exactly the characteristic function we are seeking. The solution is readily available to be

$$f(g, \theta, \tau) = \exp\left(D(\tau) + E(\tau)\theta + i\phi g\right)$$

where D, and E satisfy

$$\begin{aligned} D'(\tau) &= i\phi\left(\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2 + \frac{\sigma_K^2}{2} + E(\tau)\sigma_K\sigma_\theta\rho\right) - \frac{\phi^2\sigma_K^2}{2} + \\ & \quad E(\tau)(\delta\bar{\theta} + (1-\gamma)\sigma_K\sigma_\theta\rho + \tilde{b}\sigma_\theta^2) + E^2(\tau)\frac{\sigma_\theta^2}{2} \end{aligned} \tag{43}$$

$$E'(\tau) = i\phi(1 + \tilde{c}\sigma_K\sigma_\theta\rho) + E(\tau)(\tilde{c}\sigma_\theta^2 - \delta) \tag{44}$$

with the initial condition that $D(0) = E(0) = 0$.

Now $P1$ can be easily obtained by fourier-inverting the characteristic function

$$P1(g_T, \theta_T, 0) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{i\phi(g-\ln(X))}}{i\phi}\right] d\phi$$

This is the desired probability needed at the boundary when the option expires. $P1$ satisfies the above expression for all maturity τ, i.e.

$$P1(g_t, \theta_t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{i\phi(g-\ln(X))+D(\tau)+E(\tau)\theta_t}}{i\phi}\right] d\phi \tag{45}$$

For $P2$ the characteristic function has the same form, except the ODE's solve

$$\begin{aligned} D'(\tau) &= i\phi\left(\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2 - \frac{\sigma_K^2}{2} + E(\tau)\sigma_K\sigma_\theta\rho\right) + E^2(\tau)\frac{\sigma_\theta^2}{2} \quad (46)\\ &\quad + E(\tau)(\delta\bar{\theta} - \gamma\sigma_K\sigma_\theta\rho + \tilde{b}\sigma_\theta^2) - \left(\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2 + \frac{\phi^2\sigma_K^2}{2}\right) \\ E'(\tau) &= (i\phi - 1)(1 + \tilde{c}\sigma_K\sigma_\theta\rho) + E(\tau)(\tilde{c}\sigma_\theta^2 - \delta) \quad (47) \end{aligned}$$

6.4 D: Generalized FFBS

In this section we describe exactly how the discretized FFBS is done.

In the case of the options data, our model is captured by the two densities $p(\theta_{t+1} \mid \theta_t)$ and $p(y_t \mid \theta_t)$. After discretization, the first density is just a reparametrized AR(1), while in the second density y_t denotes the relative prices of our four put options and the density represents our nonlinear pricing equation plus the added normal error. The parameter λ is actually in both densities but we suppress it in this section in order to focus on the drawing of θ. In addition, we specify a normal prior on the initial state θ_0.

In the case of the consumption data, things are a little more complicated. The parameter ρ introduces dependence between the errors in the equation for the state evolution and equation for the observed consumption growth. To capture this, we think of the model as being expressed by the three densities,

$$p(\theta_0), \;\; p(\theta_{t+1} \mid \theta_t), \;\; p(y_{t+1} \mid \theta_{t+1}, \theta_t)$$

The third density is derived by considering the conditional distribution of the error term in the consumption growth equation, given the error term in state evolution equation which may be considered to be observed conditional on (θ_{t+1}, θ_t). Given the bivariate normality of the errors, this conditonal is easily determined.

We now give the algorithm for the general case represented by the three densities above. Obvious simplifications obtain in the case of the options data where $p(y_{t+1} \mid \theta_{t+1}, \theta_t) = p(y_{t+1} \mid \theta_{t+1})$.

Let D_t denote observations up to and including time t: $D_t = (y_1, y_2, \ldots, y_t)$. For the forward filtering, we describe how to recursively compute $p(\theta_t, \theta_{t-1} \mid D_t)$. We do this in two steps. First,

$$\begin{aligned} p(\theta_{t-1}, \theta_t, \theta_{t+1} \mid D_t) &= p(\theta_t, \theta_{t-1} \mid D_t) p(\theta_{t+1} \mid \theta_t, \theta_{t-1}, D_t) \\ &= p(\theta_t, \theta_{t-1} \mid D_t) p(\theta_{t+1} \mid \theta_t) \end{aligned}$$

Both densities are asummed available. From this joint density, we can then compute the marginal distribution $p(\theta_{t+1}, \theta_t \mid D_t)$ which we use in our second step. Note that computation of the marginal is conceptually trivial because θ is discrete.

Our second step uses,

$$\begin{aligned} p(\theta_{t+1}, \theta_t \mid D_{t+1}) &\propto p(\theta_{t+1}, \theta_t, y_{t+1} \mid D_t) \\ &= p(\theta_{t+1}, \theta_t \mid D_t) p(y_{t+1} \mid \theta_{t+1}, \theta_t, D_t) \\ &= p(\theta_{t+1}, \theta_t \mid D_t) p(y_{t+1} \mid \theta_{t+1}, \theta_t) \end{aligned}$$

Again, both of these last two densities are available and θ is discrete so we can compute $p(\theta_{t+1}, \theta_t \mid D_{t+1})$. We have have transited from $p(\theta_t, \theta_{t-1} \mid D_t)$ to $p(\theta_{t+1}, \theta_t \mid D_{t+1})$ and this is the FF recursion.

We now discuss BS (backwards sample). We wish to draw from the joint distribution $p(\theta_1, \theta_2, \ldots, \theta_T \mid D_T)$ where T represents the time of our final observation. Clearly,

$$p(\theta_1, \theta_2, \ldots, \theta_T \mid D_T) = p(\theta_T, \theta_{T-1} \mid D_T) \prod_{t=T-2}^{1} p(\theta_t \mid \theta_{t+1}, \ldots, \theta_T, D_T)$$

The key observation in FFBS is then that

$$p(\theta_t \mid \theta_{t+1}, \ldots, \theta_T, D_T) = p(\theta_t \mid D_t, \theta_{t+1}, y_{t+1})$$

because θ_t is independent of $(\theta_{t+2}, \ldots, \theta_T, y_{t+2}, \ldots, y_T)$ conditional on (y_{t+1}, θ_{t+1}). Finally, this distribution is easily computed from information obtained in our forward recursion since we have $p(\theta_t, \theta_{t+1} \mid D_{t+1} \equiv (D_t, y_{t+1}))$.

Table 1: This table reports the parameter estimates of the model by using Metropolis within Gibbs algorithm. We run the Gibbs sampler for 30,000 iterations and the first 20,000 are dropped for burn-in. The 10,000 remaining are used for all inference purposes. The data for relative put option prices are from put options on the S&P500 Future Index. We choose monthly option price (on the third Friday of every month) with 4 strike prices - ATM, 2% OTM, 3% OTM and 5% OTM with maturity 28-35 days. Data period for options is 1988-2006. The consumption data is annual and comes from BLS of non-durables and services and is deflated by the GDP deflator to convert into real consumption and further divided through by a population series to make it per capital real consumption. Data period is 1929-2006. All quantities rates are reported annually.

	Options					
	Tight			Loose		
	Posterior Median	2.5-97.5 Quantile		Posterior Median	2.5-97.5 Quantile	
σ_K	0.049	0.045	0.050	0.099	0.097	0.100
σ_θ	0.005	0.005	0.006	0.019	0.013	0.026
ρ	0.936	0.880	0.950	0.933	0.867	0.949
δ	0.096	0.079	0.099	0.486	0.351	0.500
$\bar{\theta}$	0.036	0.026	0.039	0.029	0.014	0.039
μ	-2.032	-2.000	-2.192	-2.281	-4.862	-2.008
γ	2.645	2.014	4.988	4.813	2.078	9.789
Pricing Error (σ)	0.005	0.004	0.006	0.002	0.002	0.003
	Consumption					
ψ				0.900	0.623	0.949
β				0.018	0.002	0.043

Table 2: The following table reports the long-run risk and total premia in high or low states of θ_t. We take the full posterior estimates of parameters reported in Table 1 and evaluate the posterior estimate of risk-premia given in (17) along with the long-run risk component of the risk-premia given by (35) at different levels of θ_t. The median of the posterior estimate of risk-premia and long-run risk is reported here at different levels of θ_t.

	Options					
	Tight			Loose		
	Posterior Median	2.5-97.5 Quantile		Posterior Median	2.5-97.5 Quantile	
Transient ($\gamma\sigma_K^2$)	0.006	0.005	0.0120	0.0584	0.0221	0.1042
Long Run Risk						
Constant ($\tilde{b}\sigma_K\sigma_\theta\rho$)	-0.078	-0.097	-0.055	-0.043	-0.131	0.068
Stochastic ($\tilde{c}\sigma_K\sigma_\theta\rho$)	2.771	2.444	3.048	5.753	4.428	8.419

θ	-0.05	-0.03	-0.02	-0.01	0.00	0.01	0.02	0.03	0.05
Tight	0.217	0.161	0.133	0.106	0.078	0.050	0.023	-0.005	-0.061
Loose	0.366	0.216	0.115	0.058	0.00	-0.058	-0.115	-0.173	-0.288

Table 3: This table shows the probability that a put option finishes in the money $(1-P2)$ under high and low states of θ_t. We take the median posterior parameter estimates from the data and compute the posterior probability $(1-P2)$ that a put option finishes in the money under different level of moneyness. The maturity has been fixed to 1 month for all options. The system of ODE's that $P2$ satisfies is given by (46).

Tight							
$Z(\theta_t, .08)$	1.001	1.001	1.000	1.000	0.999	0.999	0.999
$\frac{X}{R}$	θ						
	-0.03	-0.02	-0.01	0.00	0.01	0.02	0.03
1.00	0.577	0.557	0.536	0.514	0.498	0.473	0.452
0.98	0.091	0.082	0.074	0.067	0.060	0.055	0.049
0.96	0.002	0.001	0.001	0.001	0.001	0.001	0.001
0.95	0.001	0.000	0.000	0.000	0.000	0.000	0.000
Loose							
$Z(\theta_t, .08)$	1.001	1.000	1.000	1.000	0.999	0.998	0.997
$\frac{X}{R}$	θ						
	-0.03	-0.02	-0.01	0.00	0.01	0.02	0.03
1.00	0.588	0.578	0.568	0.557	0.547	0.537	0.526
0.98	0.294	0.286	0.277	0.268	0.259	0.251	0.243
0.96	0.093	0.089	0.085	0.081	0.077	0.074	0.071
0.95	0.043	0.041	0.038	0.037	0.035	0.033	0.031

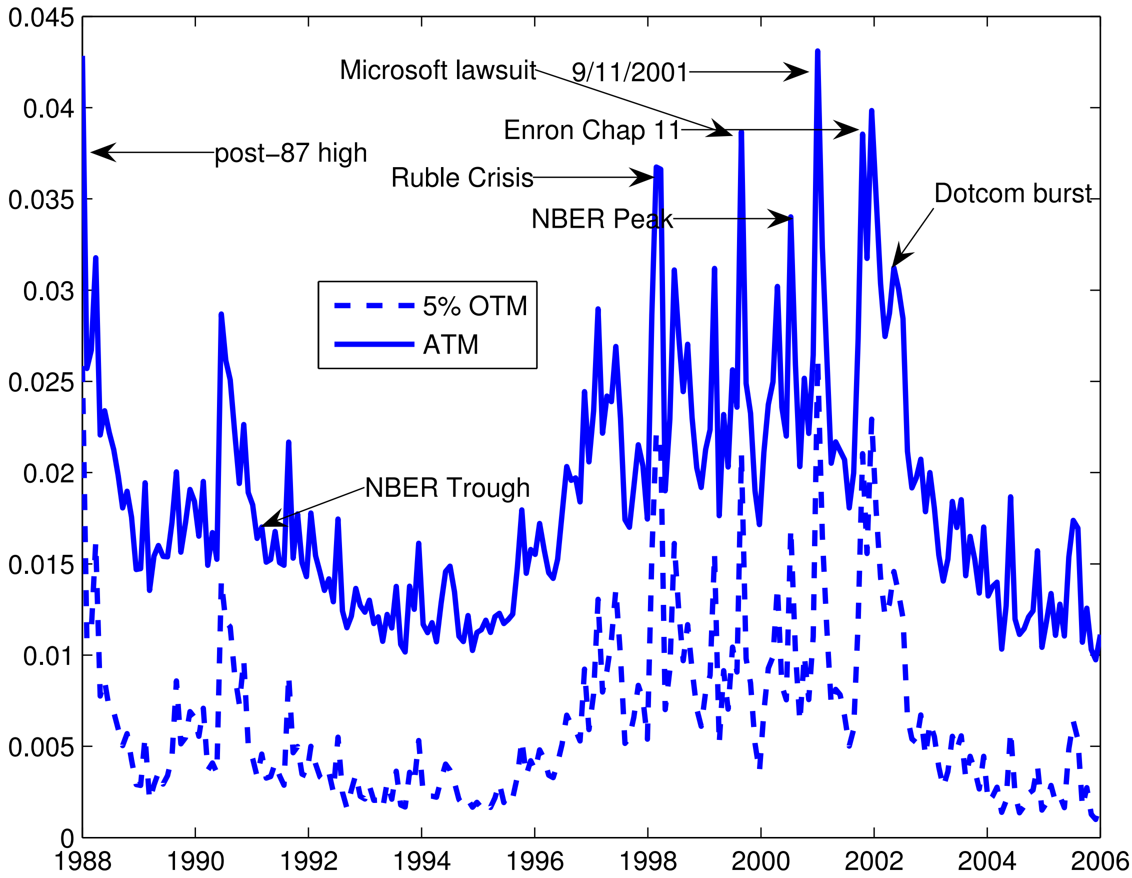


Figure 1: The Time-Series of Relative Option Prices. This is a time-series plot of put-option prices divided by the price of the underlying S&P 500 Futures for options that are at-the-money and 5% out-of-the-money, with maturity of 28-35 days. The data above is of monthly frequency and the relative price on the third Friday of every month is chosen as the representative relative price for that month when the 28-35 day maturity options are most liquid.

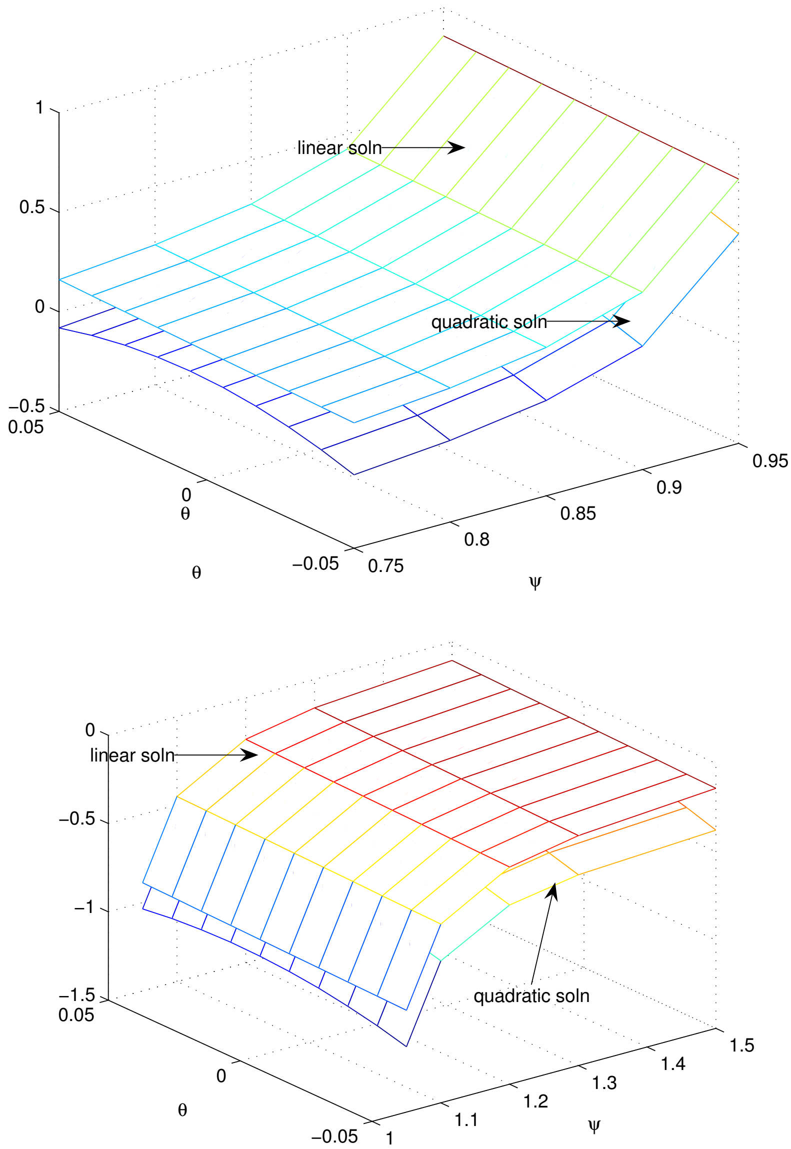


Figure 2: Plot of the left hand side of (27) by plugging in the quadratic and linear approximation of $H(\theta)$ derived in Appendix A. The top graph plots the left hand side of (27) for $\psi < 1$ over a reasonable range of the growth rates θ and the bottom graph plots the same for $\psi > 1$ across the same range of θ. The parameters used for this simulation are $\mu = -2, \gamma = 6, \beta = 0.01, \delta = 0.45, \bar{\theta} = 0.03, \sigma_\theta = .02, \sigma_K = .1, \rho = .9$.

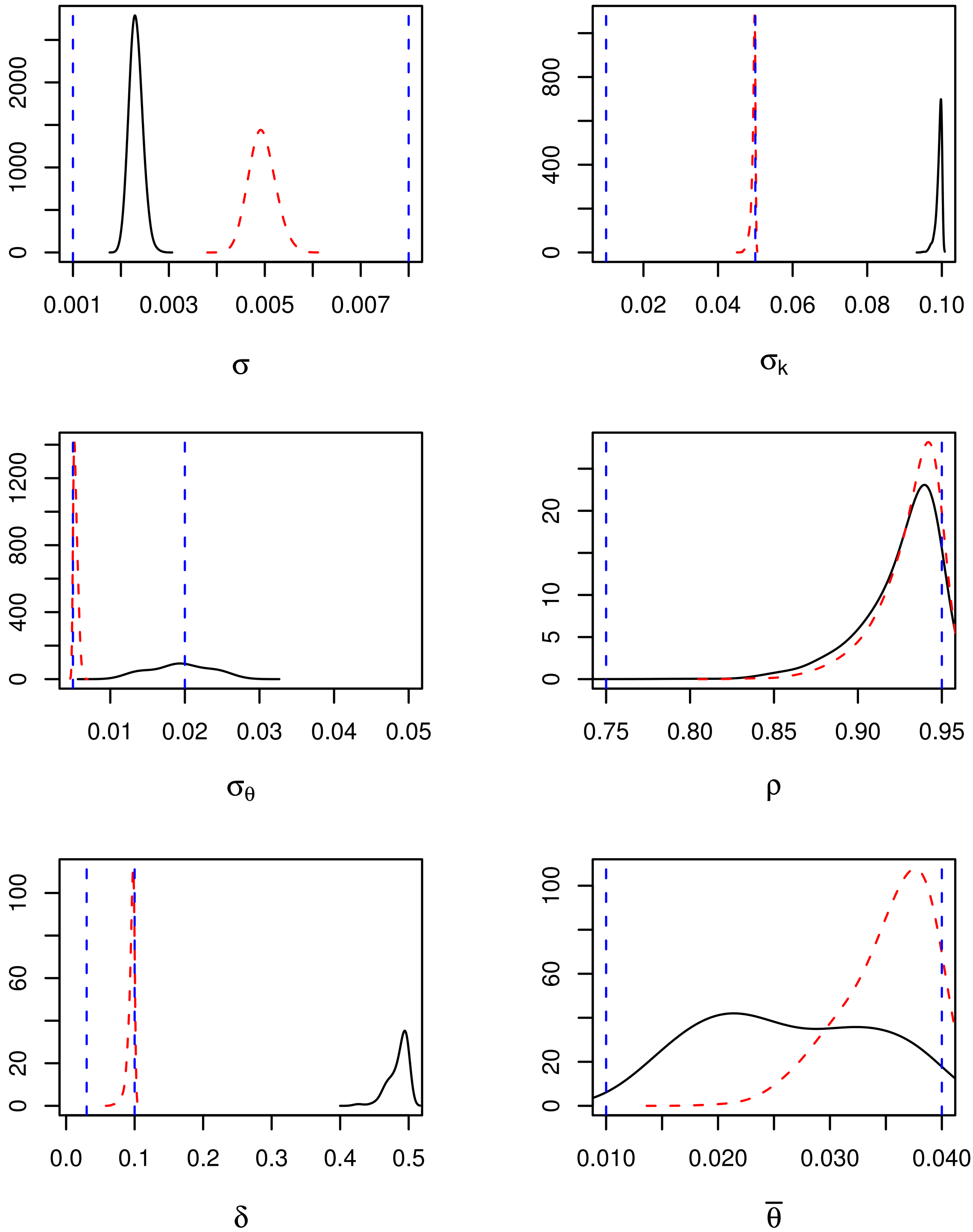


Figure 3: Priors and Posteriors of Model Parameters. The loose prior is uniform over the horizontal range of the plot. The tight prior is uniform on the range given by the two horizontal lines. The solid curve is the posterior under the loose prior. The dashed curve is the posterior under the tight prior.

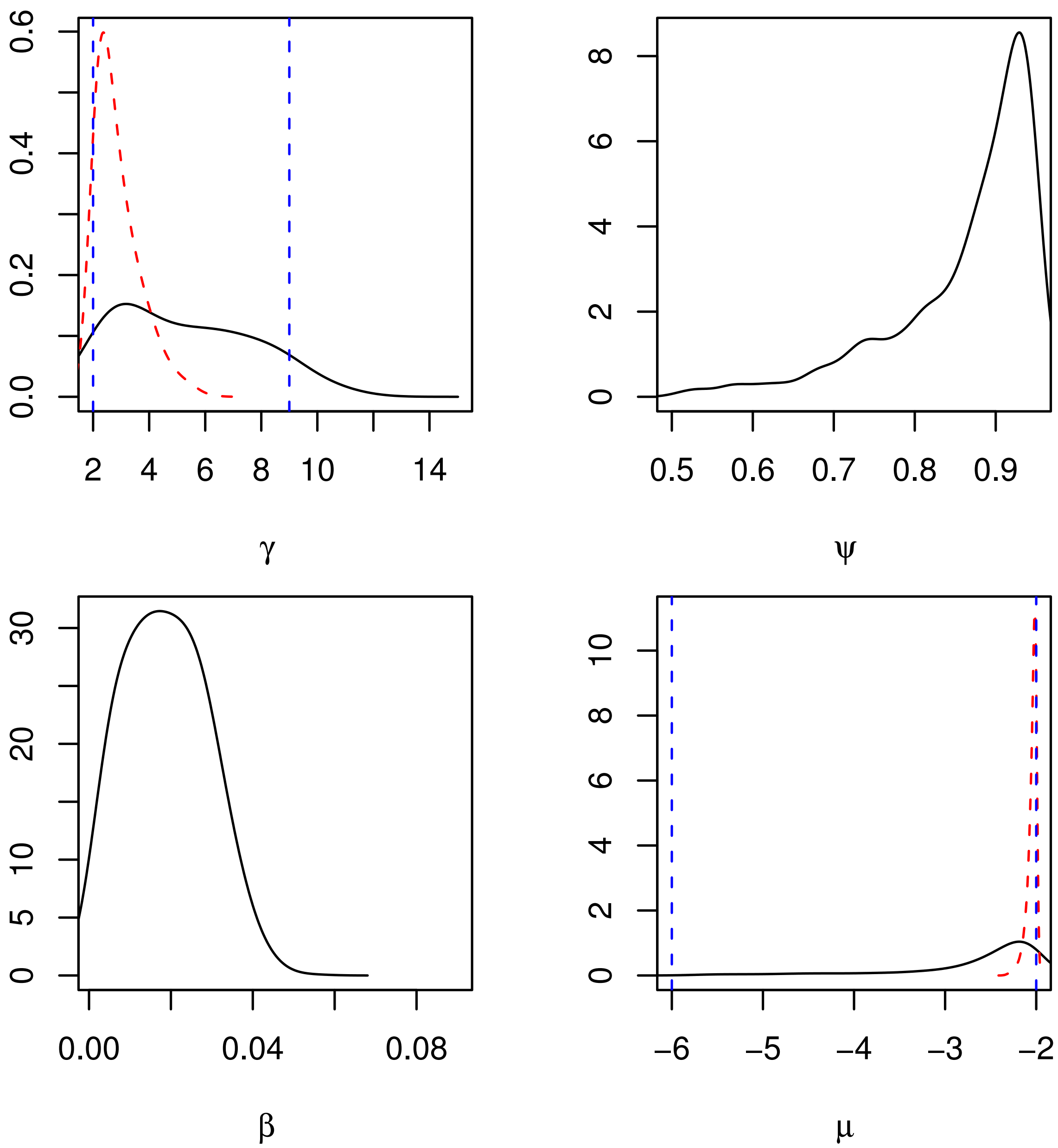


Figure 4: Priors and Posteriors of Model Parameters.The loose prior is uniform over the horizontal range of the plot. The tight prior is uniform on the range given by the two horizontal lines. The solid curve is the posterior under the loose prior. The dashed curve is the posterior under the tight prior.

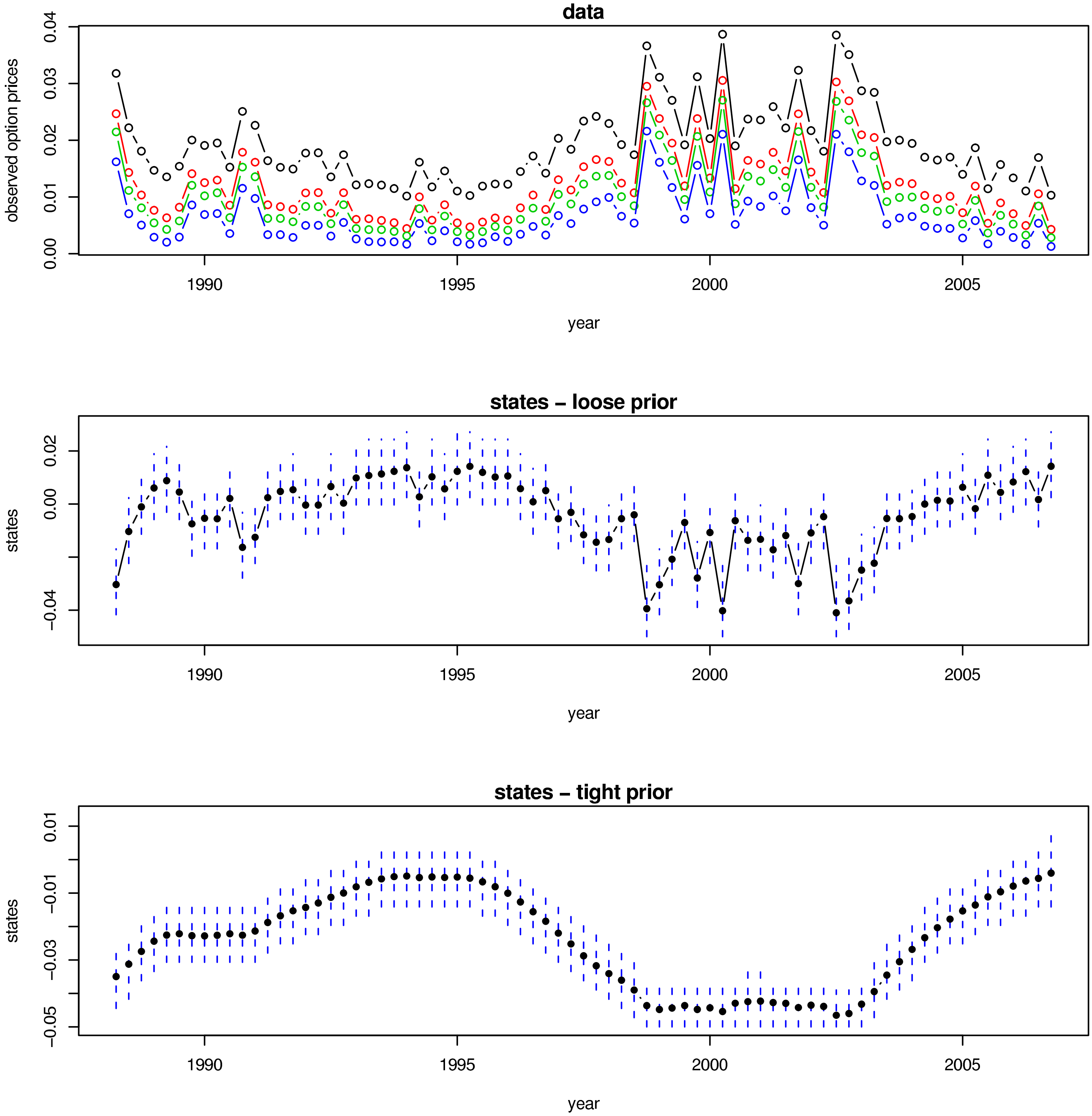


Figure 5: The top panel shows the time-series plot of the four option prices. The next two panels show the posterior distribution of the time-series of the underlying state θ_t corresponding to the loose and tight prior settings. The dashed line represents the 95% posterior band around each point of θ_t.

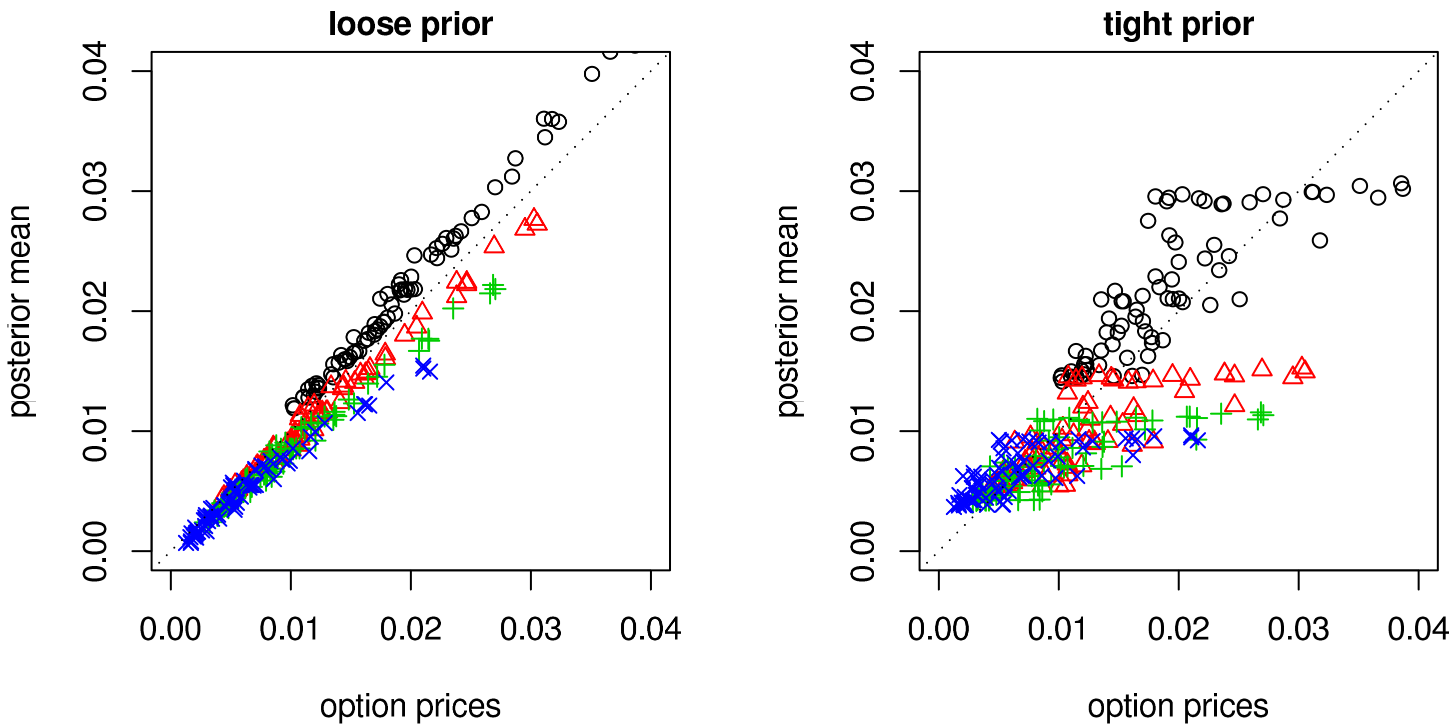


Figure 6: Scatter plot of observed option prices vs option prices implied by the model. Model implied prices are posterior means of option prices that we estimate under the two different prior settings. The four plot symbols indicate the four different types of moneynesses X/R that we consider in our analysis.

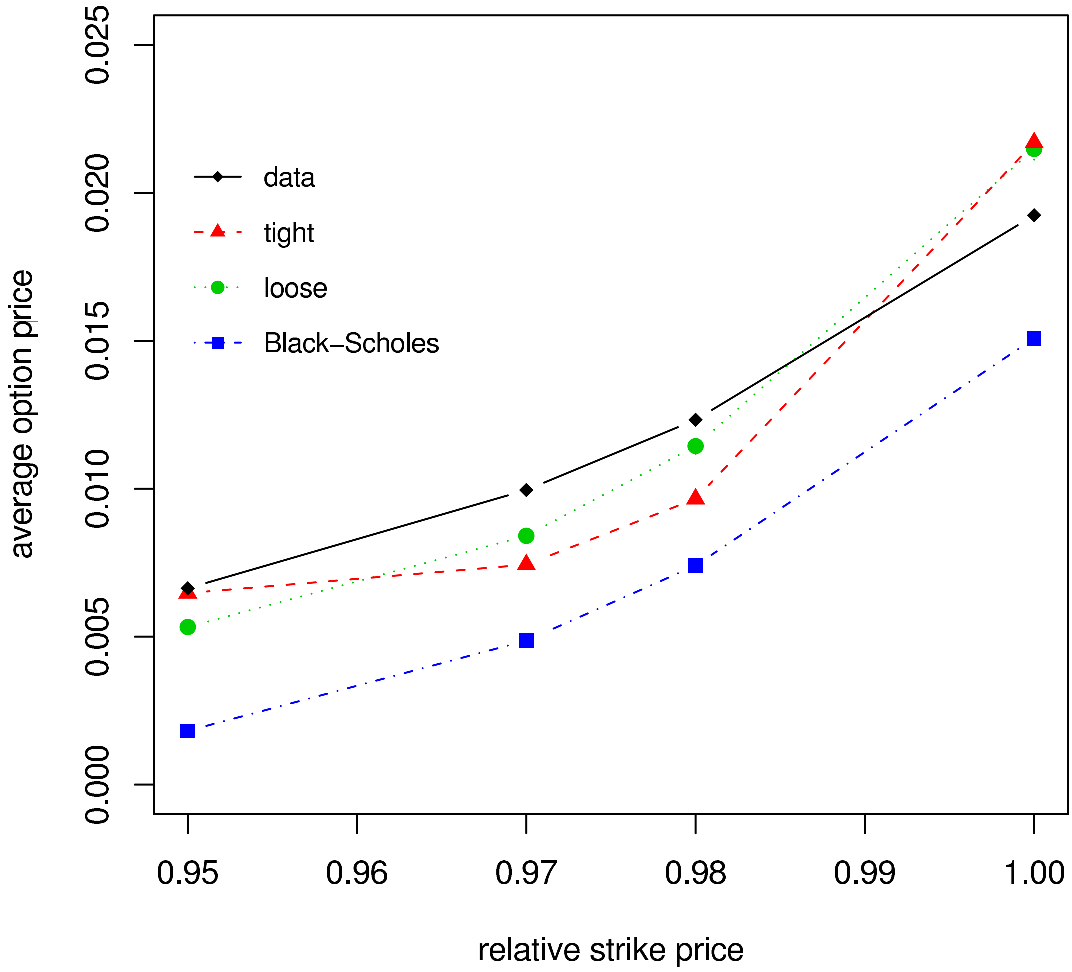


Figure 7: Cross-Sectional Fit for Put Options. The Black-Scholes line represents Black-Scholes time-series averages with $r = .0396$ (average nominal 1 month annualized interest rate from 1988-2006) and volatility of 0.15, which is the average annual volatility of the S&P 500 Index from 1988-2006. The tight and loose specifications are the averages of the time-series estimates.

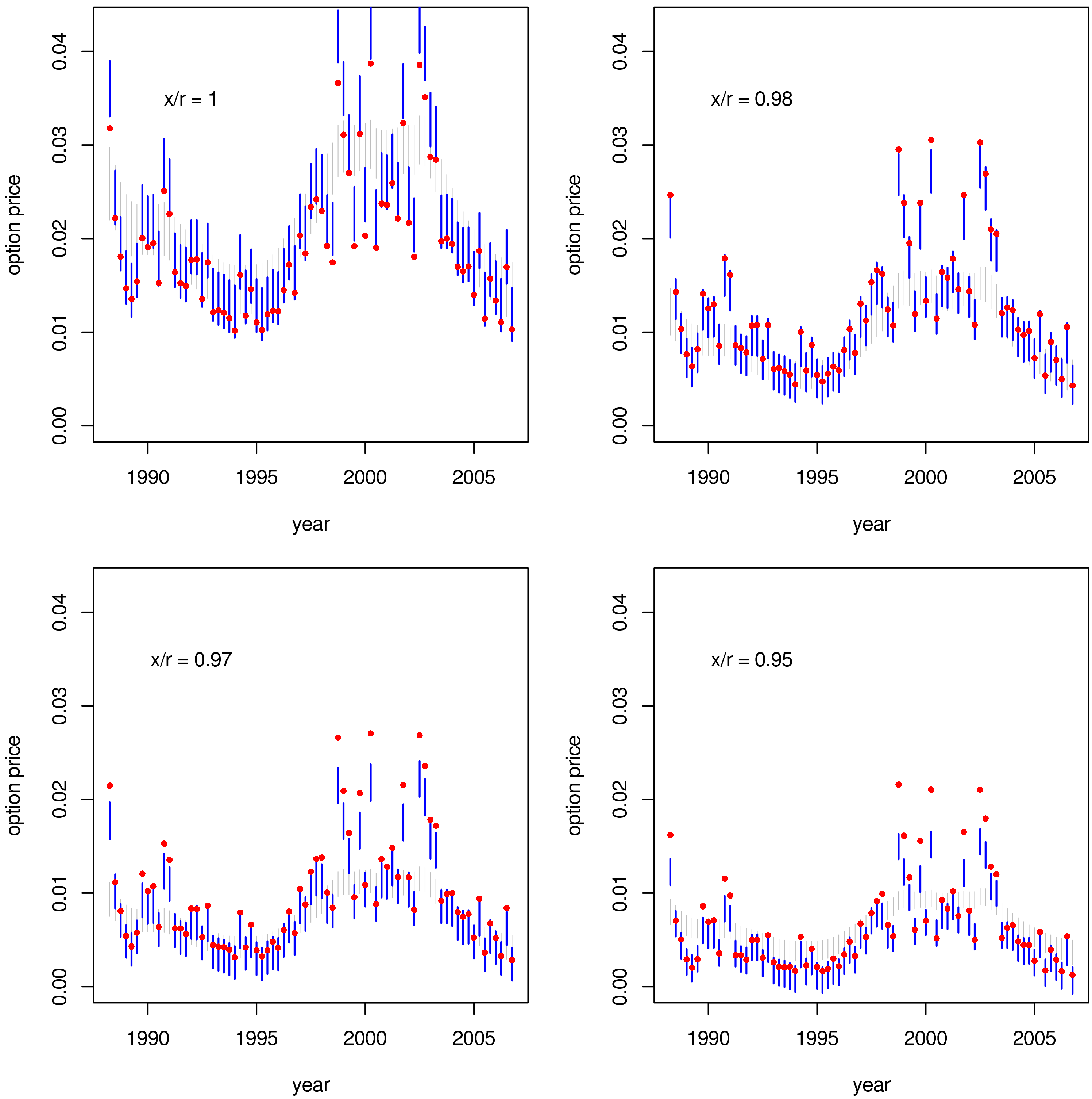
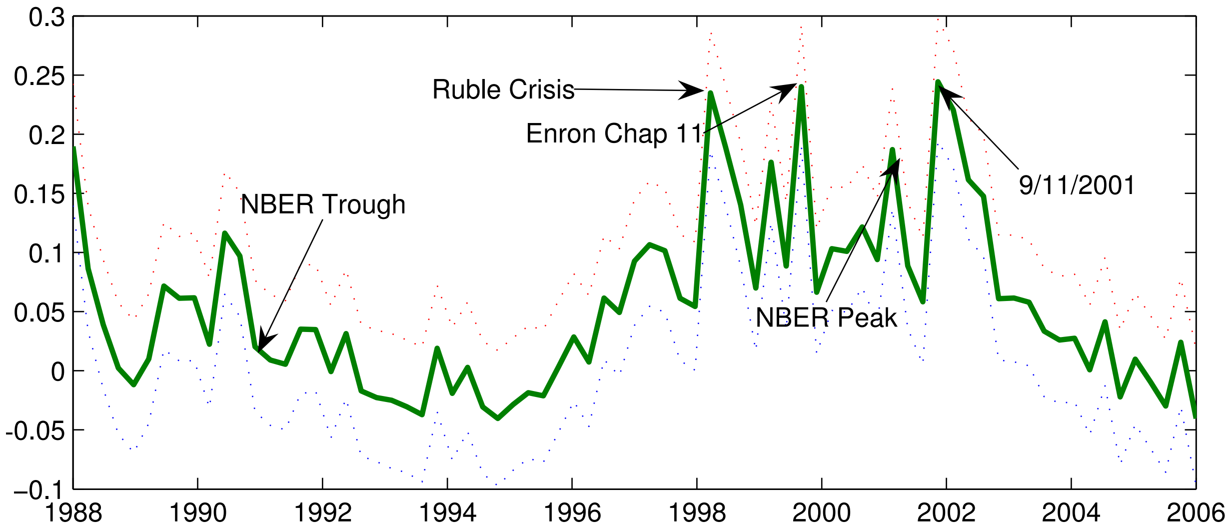


Figure 8: Time-series fit for Options. $\frac{X}{R} = 1$ implies ATM and $\frac{X}{R} = .95$ implies 5% OTM. The posterior parameter draws along with the posterior distribution of the states is used to compute the distribution of option prices at each point. The actual data is marked with a circle. The posterior distribution at each point corresponding to the loose prior is the dark band, whereas the posterior distribution corresponding to the tight prior is the light band.



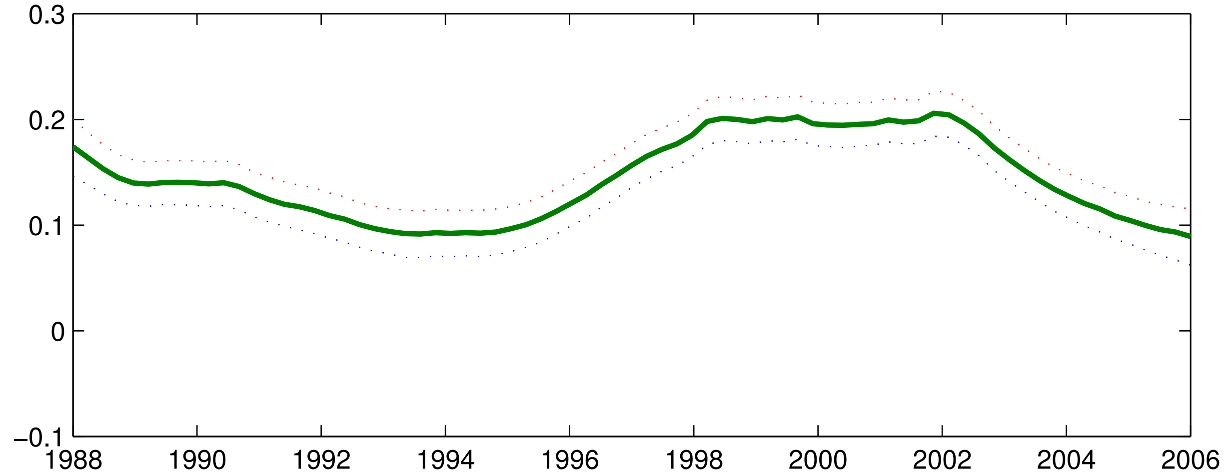


Figure 9: Time-Series of long-run risk estimated from option prices. Using the posterior distribution of parameters and the state, we compute the time-series of long-run risk given by $-\frac{H'(\theta)}{H(\theta)}\sigma_K\sigma_\theta\rho = -(\tilde{b}+\tilde{c}\theta)\sigma_K\sigma_\theta\rho$. The median time-series estimate is presented in the above plot with the dotted line representing the 95% posterior interval at each point. The top time-series plot corresponds to loose prior, whereas the bottom time-series plot corresponds to the tight prior.

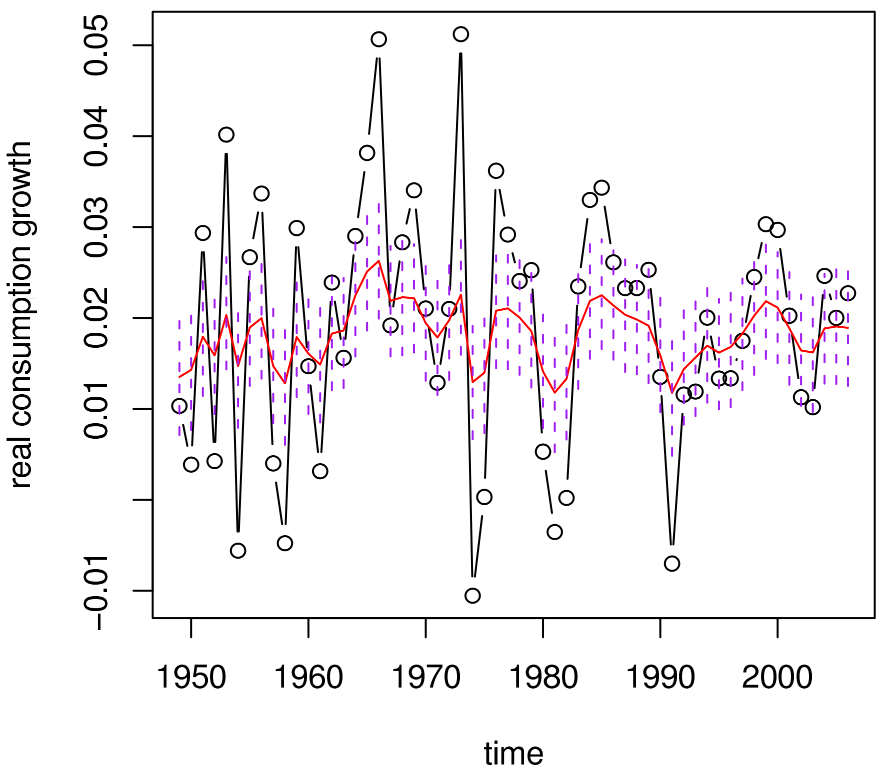


Figure 10: The black circles are the data points for aggregate consumption growth. The red line through the consumption data is the simulated posterior mean of consumption growth (μ_C) according to the model and the purple dots are the inter-quartile range of the time-series of μ_C. The priors on the parameter space correspond to the loose prior setting.

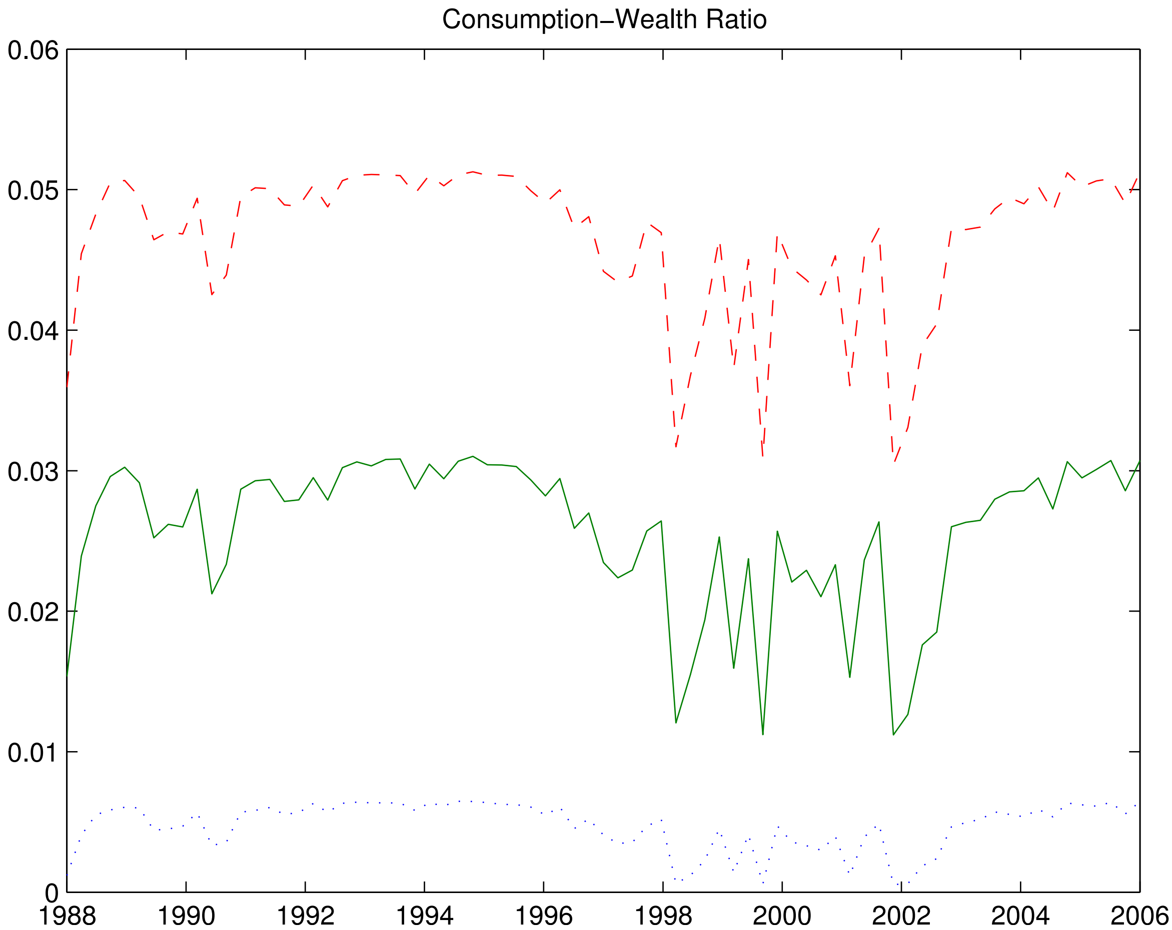


Figure 11: Time-series plot of the 2.5-97.5% confidence interval of the consumption to capital ratio in proposition 1 - $\frac{C}{K}_t = \beta^{\psi} H(\theta_t)^{\frac{1-\psi}{1-\gamma}}$. Taking the full posterior distribution of the states and the parameters from the loose prior setting, we compute the posterior distribution of the consumption to capital ratio. The dotted and dashed line represents the 2.5th and 97.5th quantile of the posterior distribution of the consumption to capital ratio, whereas the solid line is the posterior median.