Bayesian GARCH modeling

Modeling Petrobrás' log-returns

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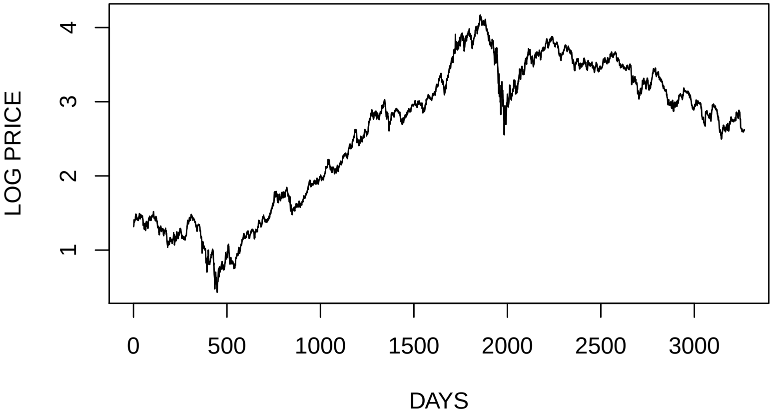
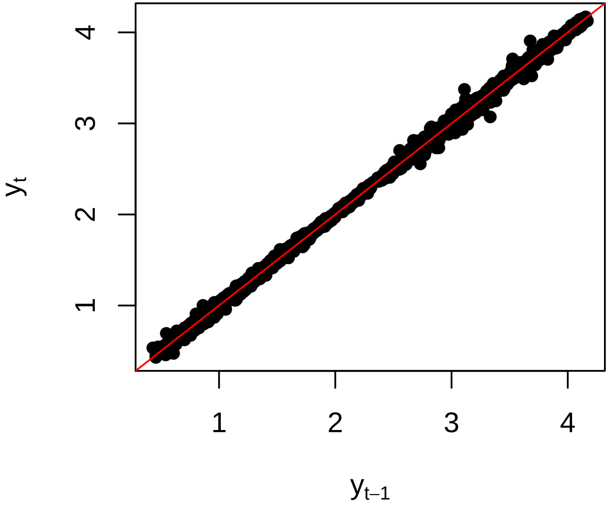
Log price: $y_t = \log p_t$

Time span: 12/29/2000 - 12/31/2013 ($n = 3268$ days)

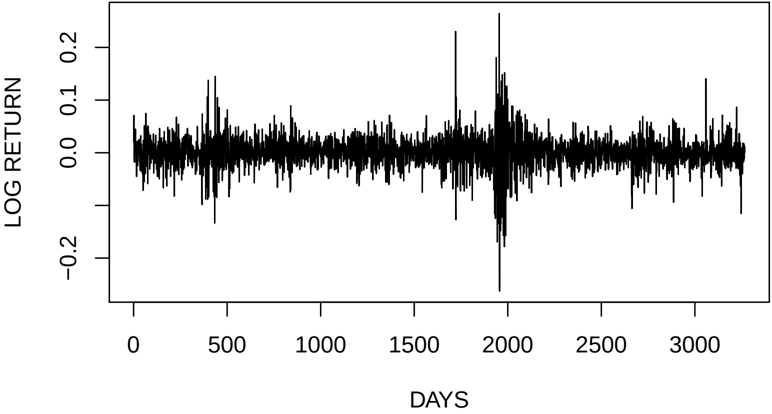


Scatterplot of y_{t-1} versus y_t

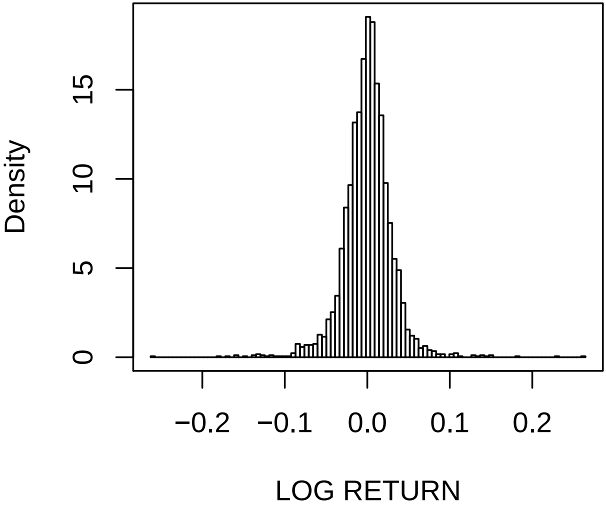


Log return: $r_t = y_t - y_{t-1} = \log(p_t/p_{t-1})$

Time span: 01/02/2001 - 12/31/2013 ($n = 3267$ days)



Histogram of r_t



Three simple models

Years 2001-2006:
The first $n_0 = 1506$ days are used for prior specification.

Years 2007-2013:
The last $n = 1760$ days are used for posterior inference.

- Model $\mathcal{M}_0$: $r_1, \ldots, r_n$ iid $N(0, \sigma^2)$
- Model $\mathcal{M}_1$: $r_1, \ldots, r_n$ iid $N(\mu, \sigma^2)$
- Model $\mathcal{M}_2$: For $t = 2, \ldots, n$

$$y_t | y_{t-1} \sim N(\alpha + \beta y_{t-1}, \sigma^2),$$

for $\alpha, \beta \in \Re$ and $\sigma^2 > 0$.

Prior

Training sample: $n_0 = 1506$ days (2001-2006)
$\tilde{y}_t$ and $\tilde{r}_t$: log prices and log returns.
$m_{\tilde{r}}$ and $V_{\tilde{r}}$: sample mean and the sample variance of r_ts.
$m_{\tilde{r}^2}$: sample mean of r_t^2s.

$$
\begin{aligned}
(m_{\tilde{r}}, V_{\tilde{r}}) &= (0.00112605, 0.0006609406) \\
m_{\tilde{r}^2} &= 0.0006617694
\end{aligned}
$$

Also, we fit

$$y_t | y_{t-1} \sim N(\alpha + \beta y_{t-1}, \sigma^2),$$

by OLS and obtain estimates (a, b, s) of (α, β, σ). Let s_a and s_b be the OLS standard errors of a and b. Therefore,

$$
\begin{aligned}
(a, b, s) &= (0.0005451564, 1.0003342482, 0.02569926) \\
(s_a, s_b) &= (0.0018491322, 0.0009933758)
\end{aligned}
$$

Model $\mathcal{M}_0$:

$$\sigma^2 \sim IG\left(5, 4m_{\tilde{r}^2}\right).$$

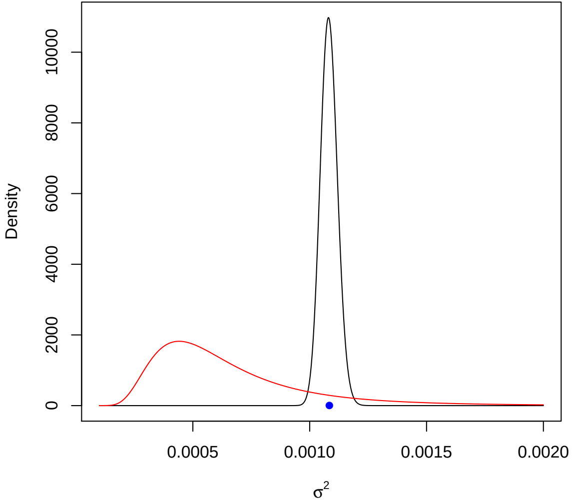
Model $\mathcal{M}_1$:

$$\begin{aligned} \mu &\sim N(m_{\tilde{r}}, 100V_{\tilde{r}}/(n_0 - 1)) \\ \sigma^2 &\sim IG(5, 4V_{\tilde{r}}) \end{aligned}$$

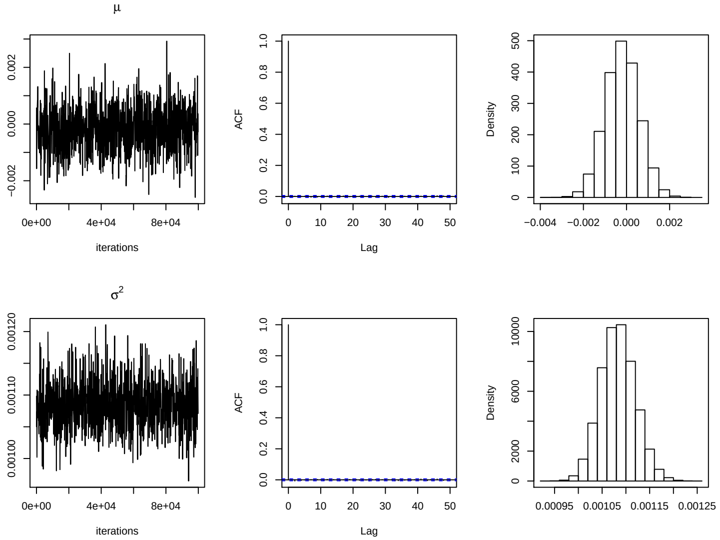
Model $\mathcal{M}_2$:

$$\begin{aligned} \alpha &\sim N(a, 100s_a^2) \\ \beta &\sim N(b, 100s_b^2) \\ \sigma^2 &\sim IG(5, 4s^2) \end{aligned}$$

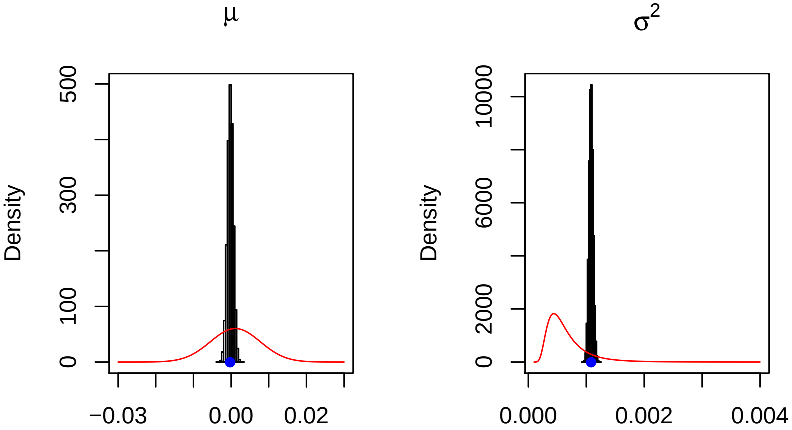
Model $\mathcal{M}_0$: Prior, posterior, MLE



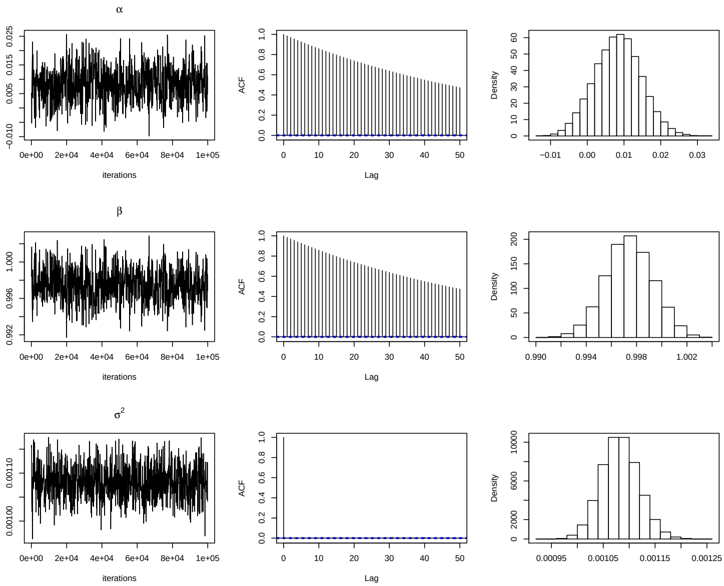
Model $\mathcal{M}_1$: Gibbs sampler output



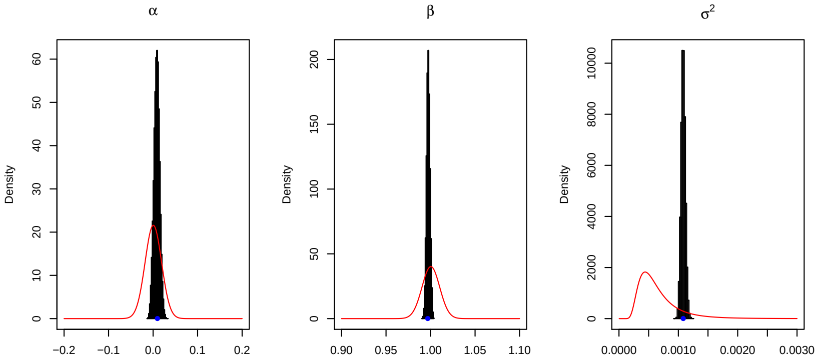
Model $\mathcal{M}_1$: Prior, posterior, MLE



Model $\mathcal{M}_2$: Gibbs sampler output



Model $\mathcal{M}_2$: Prior, posterior, MLE



$\mathcal{M}_3$: GARCH(1,1) with t errors

The GARCH(1,1) model with Student-t innovations:

$$\begin{aligned} r_t &\sim t_\nu(0, \rho h_t) \\ h_t &= \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1}, \end{aligned}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\beta > 0$.

We set the initial variance to $h_0 = 0$ for convenience.

We let $\rho = (\nu - 2)/\nu$ so that

$$V(r_t | h_t) = \frac{\nu}{\nu - 2} \rho h_t = h_t.$$

Prior

Let $\psi = (\alpha', \beta, \nu)'$ and $\alpha = (\alpha_0, \alpha_1)'$.

The prior distribution of ψ is such that

$$p(\alpha, \beta, \mu) = p(\alpha)p(\beta)p(\nu)$$

where

$$\begin{aligned} \alpha &\sim N_2(\mu_\alpha, \Sigma_\alpha) I_{(\alpha>0)} \\ \beta &\sim N(\mu_\beta, \Sigma_\beta) I_{(\beta>0)} \end{aligned}$$

and

$$p(\nu) = \lambda \exp\{-\lambda(\nu - \delta)\} I_{(\lambda>\delta)}$$

for $\lambda > 0$ and $\delta \geq 2$, such that $E(\nu) = \delta + 1/\lambda$.

Normal case: $\lambda = 100$ and $\delta = 500$.

bayesGARCH

`bayesGARCH`: Bayesian Estimation of the GARCH(1,1) Model with Student-t Innovations

```
bayesGARCH(r,mu.alpha = c(0,0),Sigma.alpha=1000*diag(1,2),
           mu.beta=0,Sigma.beta=1000,
           lambda=0.01,delta=2,control=list())
```

Paper: Ardia and Hoogerheide (2010) Bayesian Estimation of the GARCH(1,1) Model with Student-t Innovations. *The R Journal*, **2**,41-47.

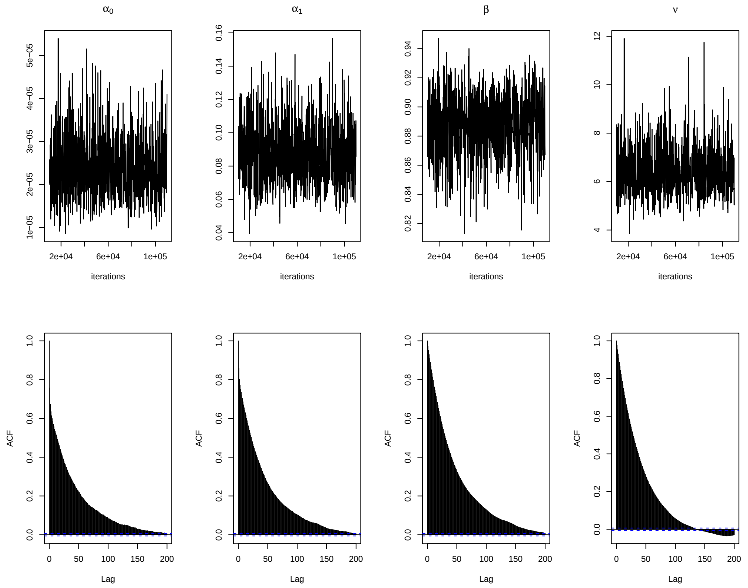
`http://cran.r-project.org/web/packages/bayesGARCH`

Example of R script

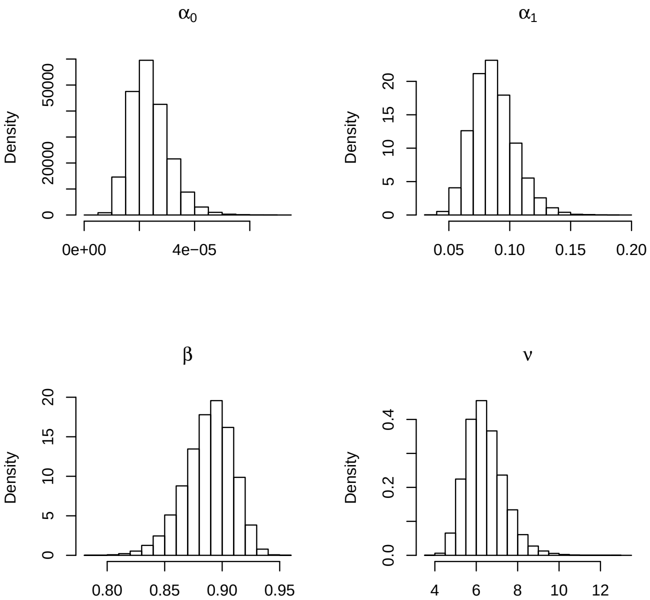
Recall that r_0 are Petrobras' returns for the first part of the data.

```
M0    = 10000       # to be discarded (burn-in)
M     = 10000       # kept for posterior inference
niter = M0+M

MCMC.initial = bayesGARCH(r0,mu.alpha=c(0,0),Sigma.alpha=1000*diag(1,2),
                          mu.beta=0,Sigma.beta=1000,lambda=0.01,delta=2,
                          control=list(n.chain=1,l.chain=niter,refresh=100))

draws = MCMC.initial$chain1

range = (M0+1):niter

par(mfrow=c(2,2))
ts.plot(draws[range,1],xlab="iterations",main=expression(alpha[0]),ylab="")
ts.plot(draws[range,2],xlab="iterations",main=expression(alpha[1]),ylab="")
ts.plot(draws[range,3],xlab="iterations",main=expression(beta),ylab="")
ts.plot(draws[range,4],xlab="iterations",main=expression(nu),ylab="")
```

Model $\mathcal{M}_3$: MCMC output

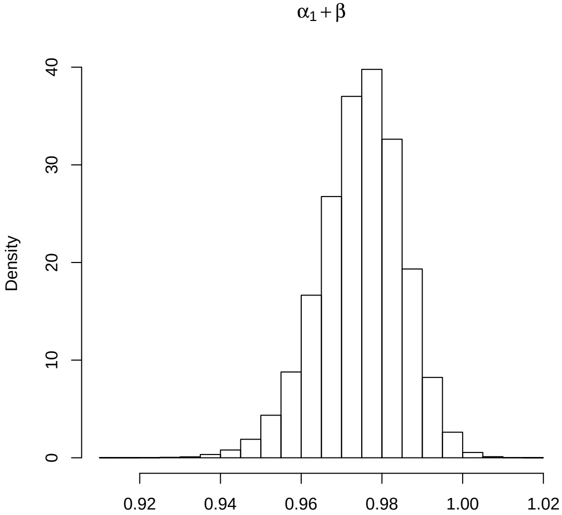


Model $\mathcal{M}_3$: Marginal posterior distributions



Model $\mathcal{M}_3$: $p(\alpha_1 + \beta | \texttt{data})$

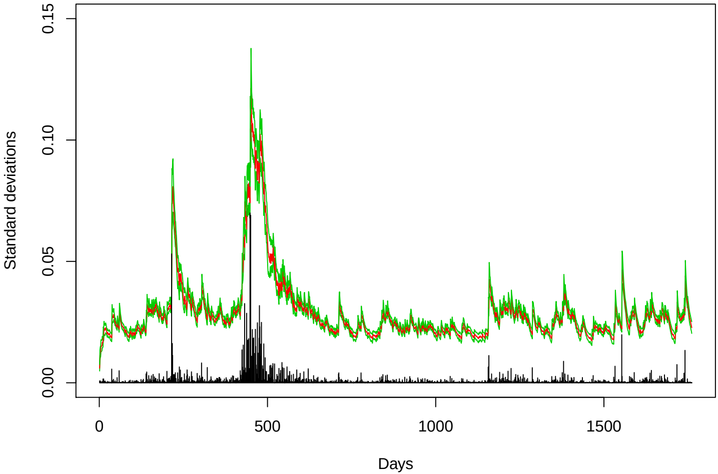
$$Pr(\alpha_1 + \beta > 1 | \texttt{data}) = 0.0034$$



Model $\mathcal{M}_3$: Quantiles from $p(h_t^{1/2}|\texttt{data})$

Percentiles 2.5%, 50% and 97.5% of $p(h_t^{1/2}|\texttt{data})$

Black vertical lines: r_t^2



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Vrontos, Dellaportas and Politis (2000) Full Bayesian Inference for GARCH and EGARCH Models. *Journal of Business & Economic Statistics*, **18**(2), 187-198.

Asai (2006) Comparison of MCMC methods for estimating GARCH models. *Journal of the Japanese Statistical Society*, **36**(2), 199-212.

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