ARCH-like models

- ARCH model
- GARCH model
- GARCH *in the mean* (GARCH-M) model
- Exponential GARCH (EGARCH) model
- Threshold GARCH (TGARCH) model

ARCH and GARCH models

Engle's (1982) autoregressive conditional heteroscedastic model, ARCH(m), assumes that

$$a_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2,$$

where $\{\epsilon_t\}$ is a sequence of independent and identically distributed (iid) random variables with mean zero and variance 1.

Bollerslev's (1986) generalized ARCH model, GARCH(m, s), assumes that

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2,$$

GARCH-M

A simple GARCH(1,1)-M model can be written as

$$\begin{aligned} r_t &= \mu + c\sigma_t^2 + a_t, \qquad\qquad a_t = \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \end{aligned}$$

where μ and c are constants.

The formulation implies serial correlations in the return series r_t.

The parameter c is called the risk premium parameter. A positive c indicates that the return is positively related to its volatility.

Other specifications of risk premium have also been used in the literature, including

$$r_t = \mu + c\sigma_t + a_t$$

and

$$r_t = \mu + c\log(\sigma_t^2) + a_t$$

EGARCH

Nelson (1991) proposes the exponential GARCH (EGARCH) model.

In particular, to allow for asymmetric effects between positive and negative asset returns, he considered the weighted innovation

$$g(\epsilon_t) = \theta\epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)],$$

or

$$g(\epsilon_t) = \begin{cases} (\theta+\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0 \\ (\theta-\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0 \end{cases}$$

where θ and γ are real constants.

An EGARCH(m, s) model can be written as

$$\log(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_{s-1} B^{s-1}}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\epsilon_{t-1})$$

EGARCH(1,1)

Let us consider the simple model with order $(1, 1)$

$$(1-\alpha B)\log(\sigma_t^2) = (1-\alpha)\alpha_0 + g(\epsilon_{t-1})$$

where the ϵ_t are iid standard normal.

In this case, $E(|\epsilon_t|) = \sqrt{2/\pi}$ and the model for $\log(\sigma_t^2)$ becomes

$$(1-\alpha B)\log(\sigma_t^2) = \begin{cases} \alpha_* + (\gamma+\theta)\epsilon_{t-1} & \text{if } \epsilon_t \geq 0 \\ \alpha_* + (\gamma-\theta)(-\epsilon_{t-1}) & \text{if } \epsilon_t < 0 \end{cases},$$

where $\alpha_* = (1-\alpha)\alpha_0 - \sqrt{2/\pi}\gamma$.

An alternative form for the EGARCH(m, s) model is

$$\log(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{s} \alpha_i \frac{|a_{t-i}| + \gamma_i a_{t-i}}{\sigma_{t-i}} + \sum_{j=1}^{m} \beta_j \log(\sigma_{t-j}^2).$$

A positive a_{t-i} contributes $\alpha_i(1 + \gamma_i)|\epsilon_{t-i}|$ to the log volatility.

A negative a_{t-i} contributes $\alpha_i(1 - \gamma_i)|\epsilon_{t-i}|$ to the log volatility.

The γ_i parameter thus signifies the leverage effect of a_{t-i}.

TGARCH

Glosten, Jagannathan, and Runkle (1993) and Zakoian (1994).

A TGARCH(m, s) model assumes the form

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{s} (\alpha_i + \gamma_i N_{t-i}) a_{t-i}^2 + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^2,$$

where N_{t-i} is an indicator for negative a_{t-i}.