ARCH-like models

- ARCH model
- GARCH model
- GARCH \emph{in the mean} (GARCH-M) model
- Exponential GARCH (EGARCH) model
- Threshold GARCH (TGARCH) model
ARCH and GARCH models

Engle’s (1982) autoregressive conditional heteroscedastic model, ARCH\((m)\), assumes that

\[
a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2,
\]

where \(\{\epsilon_t\}\) is a sequence of independent and identically distributed (iid) random variables with mean zero and variance 1.

Bollerslev’s (1986) generalized ARCH model, GARCH\((m, s)\), assumes that

\[
\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2,
\]
A simple GARCH(1,1)-M model can be written as

\[ r_t = \mu + c \sigma_t^2 + a_t, \quad a_t = \sigma_t \epsilon_t, \]
\[ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \]

where \( \mu \) and \( c \) are constants.

The formulation implies *serial correlations* in the return series \( r_t \).

The parameter \( c \) is called the *risk premium* parameter. A positive \( c \) indicates that the return is positively related to its volatility.

Other specifications of risk premium have also been used in the literature, including

\[ r_t = \mu + c \sigma_t + a_t \]

and

\[ r_t = \mu + c \log(\sigma_t^2) + a_t \]
EGARCH

Nelson (1991) proposes the exponential GARCH (EGARCH) model.

In particular, to allow for asymmetric effects between positive and negative asset returns, he considered the weighted innovation

\[ g(\epsilon_t) = \theta \epsilon_t + \gamma [|\epsilon_t| - E(|\epsilon_t|)], \]

or

\[ g(\epsilon_t) = \begin{cases} 
(\theta + \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0 \\
(\theta - \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0 
\end{cases} \]

where \( \theta \) and \( \gamma \) are real constants.

An EGARCH\((m, s)\) model can be written as

\[
\log(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_{s-1} B^{s-1}}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\epsilon_{t-1})
\]
EGARCH(1,1)

Let us consider the simple model with order (1, 1)

$$(1 - \alpha B) \log(\sigma_t^2) = (1 - \alpha)\alpha_0 + g(\epsilon_{t-1})$$

where the $\epsilon_t$ are iid standard normal.

In this case, $E(|\epsilon_t|) = \sqrt{2/\pi}$ and the model for $\log(\sigma_t^2)$ becomes

$$(1 - \alpha B) \log(\sigma_t^2) = \begin{cases} 
\alpha_* + (\gamma + \theta)\epsilon_{t-1} & \text{if } \epsilon_t \geq 0 \\
\alpha_* + (\gamma - \theta)(-\epsilon_{t-1}) & \text{if } \epsilon_t < 0 
\end{cases},$$

where $\alpha_* = (1 - \alpha)\alpha_0 - \sqrt{2/\pi}\gamma$. 
An alternative form for the EGARCH\((m, s)\) model is

\[
\log(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{s} \alpha_i \frac{|a_{t-i}| + \gamma_i a_{t-i}}{\sigma_{t-i}} + \sum_{j=1}^{m} \beta_j \log(\sigma_{t-j}^2).
\]

A positive \(a_{t-i}\) contributes \(\alpha_i (1 + \gamma_i) |\epsilon_{t-i}|\) to the log volatility.

A negative \(a_{t-i}\) contributes \(\alpha_i (1 - \gamma_i) |\epsilon_{t-i}|\) to the log volatility.

The \(\gamma_i\) parameter thus signifies the leverage effect of \(a_{t-i}\).

A TGARCH\((m, s)\) model assumes the form

\[
\sigma_t^2 = \alpha_0 + \sum_{i=1}^{s} (\alpha_i + \gamma_i N_{t-i}) a_{t-i}^2 + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^2,
\]

where \(N_{t-i}\) is an indicator for negative \(a_{t-i}\).