Vector Autoregressions

Lecture 10

Bueno, 2011 - Chapter 6
Enders, 2004 - Chapter 5
Heij et al., 2004 - Chapter 7.6
Lütkepohl, 2006
Morettin, 2011 - Chapter 9
Sims, 1980

Outline

VAR(p) model

- VAR(1)
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VAR at a glance

Del Negro and Schorfheide (2011) says

"VARs appear to be straightforward multivariate generalizations of univariate autoregressive models. They turn out to be one of the key empirical tools in modern macroeconomics.

Sims (1980) proposed that VARs should replace large-scale macroeconometric models inherited from the 1960s, because the latter imposed incredible restrictions, which were largely inconsistent with the notion that economic agents take the effect of todays choices on tomorrows utility into account.

VARs have been used for macroeconomic forecasting and policy analysis to investigate the sources of business-cycle fluctuations and to provide a benchmark against which modern dynamic macroeconomic theories can be evaluated."

Meus primeiros artigos

Efeitos dinâmicos dos choques de oferta e demanda agregada sobre o nível de atividade econômica do Brasil.
Revista Brasileira de Economia (1993).

http://hedibert.org/wp-content/uploads/2013/12/lima-migon-lopes-1993.pdf

Tendência estocástica do produto no Brasil: efeitos das flutuações da taxa de crescimento da produtividade e da taxa de juro real.
Pesquisa e Planejamento Econômico (1995).

http://hedibert.org/wp-content/uploads/2013/12/lima-moreira-lopes-pereira-1995.pdf

Um modelo para a previsão conjunta do PIB, inflação e liquidez.
Revista de Econometria (1997).

http://hedibert.org/wp-content/uploads/2013/12/moreira-fiorencio-lopes-1997.pdf

Revista de Econometria (1997)

This paper analyses an empirical relationship between Brazilian GNP, inflation and liquidity using structural VEC models.

We split the model into a marginal and a conditional block, using Granger causality test for integrated variables as the separating criteria.

We estimated the marginal model in its VEC representation and could not reject the hypothesis that the marginal model has three common trends.

We used as one identifying criteria of the model the separation between the shocks that have permanent effects and the ones that only have transitory effects.

We used additional (economic theory) restrictions, to identify the structural model and to interpret each permanent shock as an exogenous change in economic policy.

We identified three shocks (real interest, liquidity and supply) and investigate their impulse-response functions.

VAR(p)

Let $y_t = (y_{t1}, \ldots, y_{tq})'$ contain q (macroeconomic) time series observed at time t.

The (basic) VAR(p) can be written as

$$y_t = \nu + A_1 y_{t-1} + \ldots + A_p y_{t-p} + u_t$$

where

$$u_t \sim \text{i.i.d. } N(0, \Sigma)$$

and

- $A_1, \ldots, A_p$ are $(q \times q)$ autoregressive matrices
- Σ is an $(q \times q)$ variance-covariance matrix

VAR(1)

For illustration, consider the VAR(1) model:

$$y_t = \nu + Ay_{t-1} + u_t.$$

As in the univariate AR(1) model, we can rewrite y_t as a function of y_0 and the error terms $u_1, u_2, \ldots, u_n$:

$$y_t = (I_q + A + A^2 + \cdots A^{t-1})\nu + A^t y_0 + \sum_{i=0}^{t-1} A^i u_{t-i}.$$

Stable VAR(1)

If all eigenvalues of A fall outside the unit circle, the sequence $\{I_q, A, A^2, A^3, \ldots\}$ is such that

$$V\left(\sum_{i=0}^{\infty} A^i u_{t-i}\right) < \infty.$$

We then say that this is an *stable VAR(1)* process.

This is equivalent to saying that

$$|I_q - Az| \neq 0 \qquad |z| \leq 1.$$

In such case, as $j \to \infty$, it follows that

- $(I_q + A + A^2 + \cdots + A^j)\nu \longrightarrow (I_q - A)^{-1}\nu \equiv \mu$,
- $A^j y_0 \longrightarrow 0$.

VMA(∞) + h-step ahead forecast

The stable VAR(1) can be rewritten as

$$y_t = \mu + \sum_{i=1}^{\infty} A^i u_{t-i}$$

and follows straightforwardly that the unconditional mean of y_t is

$$E(y_t) = \mu$$

and the autocovariance function of y_t is

$$\Gamma_y(h) = E\{(y_t - \mu)(y_{t-h} - \mu)\} = \sum_{i=0}^{\infty} A^{h+i} \Sigma (A^i)'$$

***h*-step ahead forecast:**

$$y_t(h) = (I_q + A + A^2 + \cdots + A^{h-1})\nu + A^h y_t$$

VAR(p)

A VAR(p) is stable if

$$|I_q - A_1 z - A_2 z^2 - \cdots A_p z^p| \neq 0 \qquad |z| \leq 1.$$

All stable VAR(p) processes are stationary VAR(p) processes.

Example: Trivariate VAR(1)

$$y_t = \nu + \begin{pmatrix} 0.5 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.3 \\ 0.0 & 0.2 & 0.3 \end{pmatrix} y_{t-1} + u_t.$$

Therefore,

$$\begin{aligned} \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.3 \\ 0.0 & 0.2 & 0.3 \end{pmatrix} z \right| &= \begin{vmatrix} 1-0.5z & 0.0 & 0.0 \\ -0.1z & 1-0.1z & -0.3z \\ 0.0 & -0.2z & 1-0.3z \end{vmatrix} \\ &= (1-0.5z)(1-0.4z-0.03z^2) \end{aligned}$$

The solutions of

$$(1-0.5z)(1-0.4z-0.03z^2) = 0$$

are $z_1 = 2$, $z_2 = 2.1525$ and $z_3 = -15.4858$; all outside the unit circle.

Therefore the above VAR(1) is stable $\Rightarrow$ stationary.

Example: Bivariate VAR(2)

$$y_t = \nu + \begin{pmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{pmatrix} y_{t-1} + \begin{pmatrix} 0.0 & 0.0 \\ 0.25 & 0.0 \end{pmatrix} y_{t-2} + u_t .$$

Therefore,

$$\left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{pmatrix} z - \begin{pmatrix} 0.0 & 0.0 \\ 0.25 & 0.0 \end{pmatrix} z^2 \right|$$

equals $1 - z + 0.21z^2 - 0.025z^3$.

The three solutions of $1 - z + 0.21z^2 - 0.025z^3 = 0$ are 1.3 and the pair of complex roots $3.55 \pm 4.26i$.

All roots are outside the unit circle:
$|3.55 \pm 4.26i| = \sqrt{3.55^2 + 4.26^2} = 5.545$.

Therefore the above VAR(2) is stable $\Rightarrow$ stationary.

Rewritting a VAR(p) as a VAR(1)

The VAR(p) model

$$y_t = \nu + A_1 y_{t-1} + \ldots + A_p y_{t-p} + u_t.$$

can be rewritten by stacking $y_t, y_{t-1}, \ldots, y_{t-p+1}$ on a larger vector:

$$\begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} \nu \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_q & 0 & \cdots & 0 & 0 \\ 0 & I_q & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_q & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} u_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or, more compactly, a VAR(1) model for pq time series:

$$y_t^* = \nu^* + A y_{t-1}^* + u_t^*.$$

***h*-step ahead forecast:**

$$y_t^*(h) = (I_{qp} + A + A^2 + \cdots + A^{h-1})\nu + A^h y_t^*$$

Stationarity

A VAR(p) is covariance-stationary if all values of z satisfying

$$|I_q - A_1 z - A_2 z^2 - \cdots - A_p z^p| = 0$$

lie outside the unit circle.

This is equivalent to all eigenvalues of A lying inside the unit circle.

Trivariate VAR(1): Eigenvalues of A are 0.5, 0.467 and -0.065.

Bivariate VAR(2): Eigenvalues of A are 0.769 and $0.115 \pm 0.139i$, with $|0.115 \pm 0.139i| = \sqrt{0.115^2 + 0.139^2} = 0.1804$.

Mean, autocovariance and h-step ahead forecast

The larger VAR(1) model

$$y_t^* = \nu^* + Ay_{t-1}^* + u_t^*.$$

has, if stable, unconditional mean

$$\mu^* = (I_{qp} - A)^{-1}\nu^*,$$

and autocovariance function

$$\Gamma_{y^*}(h) = \sum_{i=1}^{\infty} A^{h-i}\Sigma_{u^*}(A^i)'$$

If $J = (I_q, 0, \ldots, 0)$ of dimension $q \times qp$, if follows that

$$\begin{aligned} y_t &= Jy_t^* \\ E(y_t) &= J\mu^* \\ \Gamma_y(h) &= J\Gamma_{y^*}(h)J', \end{aligned}$$

and h-step ahead point-forecast

$$y_t(h) = Jy_t^*(h).$$

VMA(∞)

If all eigenvalues of A lie inside the unit circle, then

$$y_t = \mu + \sum_{i=0}^{\infty} \Phi_i u_{t-i}$$

where

$$\mu = (I_q - A_1 - A_2 - \cdots - A_p)^{-1}\nu$$

and

$$\begin{aligned} \Phi_0 &= I_q \\ \Phi_i &= \sum_{j=1}^{i} \Phi_{i-j} A_j \qquad \text{for } i = 1, 2, \ldots \\ \Phi_i &= 0 \quad \text{for } i < 0. \end{aligned}$$

Variance Decomposition

The mean square error (MSE) of the h-step ahead forecast is

$$\Sigma + \Phi_1 \Sigma \Phi_1' + \cdots + \Phi_{h-1} \Sigma \Phi_{h-1}'.$$

The error $u_t \sim N(0, \Sigma)$ can be orthogonalized by

$$\varepsilon_t = A^{-1} u_t \sim N(0, D)$$

where $\Sigma = ADA'$ and D is diagonal (for instance, via singular value decomposition or Cholesky decomposition).

The MSE of the h-step ahead forecast can be rewritten as the conbribution of the jth orthogonalized innovation to the MSE is

$$d_j(a_j a_j' + \Phi_1 a_j a_j' \Phi_1' + \cdots + \Phi_{h-1} a_j a_j' \Phi_{h-1}')$$

Impulse-response function

The matrix Φ_s has the interpretation

$$\frac{\partial y_{t+s}}{\partial u_s'} = \Phi_s,$$

that is, the (i, j) element of Φ_s identifies the consequences of a one-unit increase in the innovation of variable j at time t for the value of variable i at time $t + s$, holding all other innovations at all dates constant.

A plot of the (i, j) element of Φ_s,

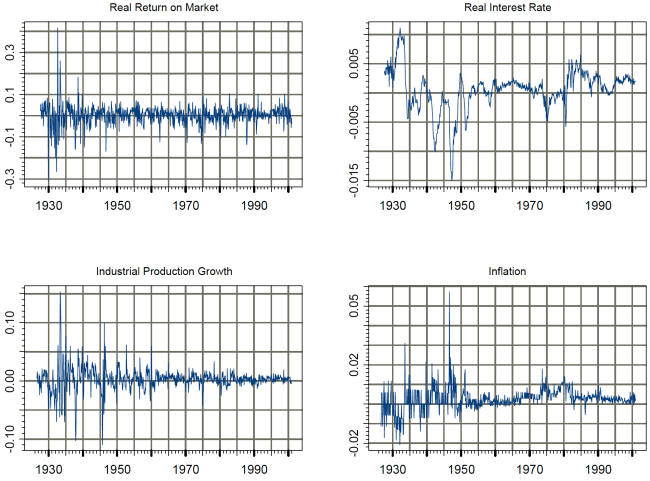
$$\frac{\partial y_{t+s,i}}{\partial u_{s,i}},$$

as a function of s is called the *impulse-response function*.

Similar to the forecast function, the impulse-response function is also highly nonlinear on $A_1, \ldots, A_q$.

Real data example

Monthly real stock returns, real interest rates, real industrial production growth and the inflation rate (1947.1 – 1987.12)



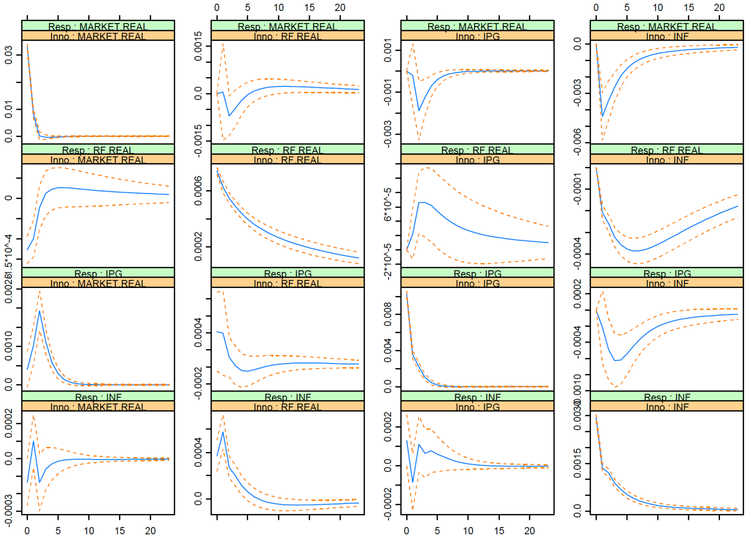
Estimates

$$\hat{\nu} = \begin{pmatrix} 0.0074 & 0.0002 & 0.0010 & 0.0019 \end{pmatrix}'$$

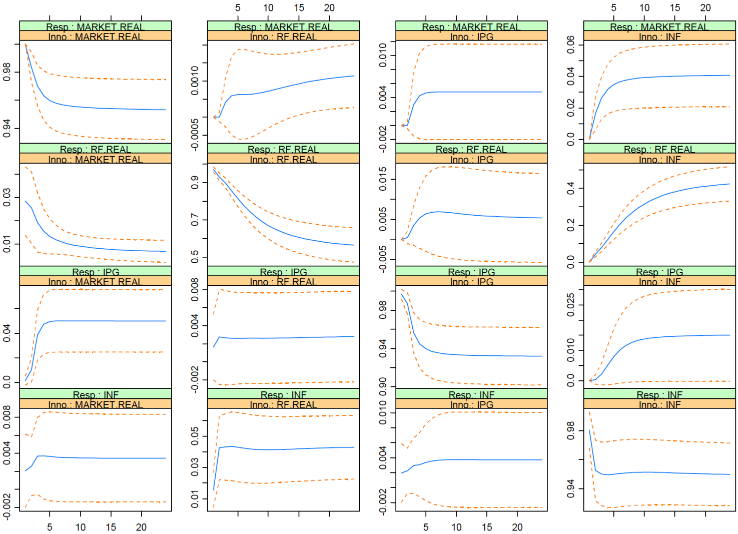
$$\widehat{A}_1 = \begin{pmatrix} 0.24 & 0.81 & -1.50 & 0.00 \\ 0.00 & 0.88 & -0.71 & 0.00 \\ 0.01 & 0.06 & 0.46 & -0.01 \\ 0.03 & 0.38 & -0.07 & 0.35 \end{pmatrix}$$

$$\widehat{A}_2 = \begin{pmatrix} -0.05 & -0.35 & -0.06 & -0.19 \\ 0.00 & 0.04 & 0.01 & 0.00 \\ -0.01 & -0.59 & 0.25 & 0.02 \\ 0.04 & -0.33 & -0.04 & 0.09 \end{pmatrix}$$

Impulse-response functions



Variance decomposition



Construção de Modelos VAR

A construção de modelos VAR segue o mesmo ciclo de

- Identificação;
- Estimação; e
- Diagnóstico,

usado para modelos univariados da classe ARMA.

Identificação

Uma maneira de identificar a ordem p de um modelo VAR(p) consiste em ajustar sequencialmente modelos autorregressivos vetoriais de ordens $1, 2, \ldots, k$ e testar a significância dos coeficientes (matrizes).

Considere os modelos

$$\begin{aligned}
VAR(1): \quad y_t &= A_1^{(1)} y_{t-1} + u_t^{(1)} \\
VAR(2): \quad y_t &= A_1^{(2)} y_{t-1} + A_2^{(2)} y_{t-2} + u_t^{(2)} \\
&\vdots \\
VAR(k): \quad y_t &= A_1^{(k)} y_{t-1} + A_2^{(k)} y_{t-2} + \cdots + A_k^{(k)} y_{t-k} + u_t^{(k)}
\end{aligned}$$

Os parâmetros podem ser estimados por MQO, que fornecem estimadores consistentes e eficientes.

Dessa forma, objetivamos testar

$$\begin{aligned} H_0 &: A_k^{(k)} = 0 \\ H_a &: A_k^{(k)} \neq 0. \end{aligned}$$

O teste da razão de verossimilhanças é baseado nas estimativas das matrizes de covariâncias dos resíduos dos modelos ajustados.

Estatística da Razão de Verossimilhanças

A estatística da razão de verossimilhanças para o teste de interesse fica dada por

$$RV(k) = (T-k)\log\left\{\frac{|\widehat{\Sigma}^{(k-1)}|}{|\widehat{\Sigma}^{(k)}|}\right\},$$

que, sob H_0, segue uma distribuição $\chi^2_{q^2}$.

A matriz de covariâncias dos resíduos, que estima Σ, é dada por

$$\widehat{\Sigma}^{(k)} = \frac{1}{T-k}\sum_{t=k+1}^{T} \hat{u}_t^{(k)}(\hat{u}_t^{(k)})'$$

Critérios de informação

Outra forma de identificar a ordem de uma VAR é usar algum critérios de informação, como:

$$
\begin{aligned}
AIC(k) &= \log|\widehat{\Sigma}^{(k)}| + \left(\frac{kq^2}{T}\right) \times 2 \\
BIC(k) &= \log|\widehat{\Sigma}^{(k)}| + \left(\frac{kq^2}{T}\right) \times \log(T) \\
HQC(k) &= \log|\widehat{\Sigma}^{(k)}| + \left(\frac{kq^2}{T}\right) \times \log(\log(T)) \\
FPE(k) &= \log|\widehat{\Sigma}^{(k)}| + q\log\left(\frac{T+qk+1}{T-qk-1}\right)
\end{aligned}
$$

Estimação

Para um determinado valor de p e assumindo que

$$u_t \sim N(0, \Sigma),$$

podemos estimar os coeficientes por máxima verossimilhança.

Neste caso, os estimadores de MV condicional serão equivalentes aos estimadores de MQO.

VAR(1)

Vamos ilustrar para o caso VAR(1). Nesse caso, os EMV condicionais são obtidos maximizando-se a seguinte função de log-verossimilhança:

$$\begin{aligned}\mathcal{L}(\theta) &= -\frac{q(T+1)}{2}\log(2\pi) + \frac{T-1}{2}\log|\Sigma^{-1}| \\ &- \frac{1}{2}\sum_{t=2}^{T}(y_t - Ay_{t-1})'\Sigma^{-1}(y_t - Ay_{t-1}),\end{aligned}$$

obtendo-se

$$\begin{aligned}\widehat{A} &= \left(\sum_{t=2}^{T} y_{t-1}y_{t-1}'\right)^{-1}\left(\sum_{t=2}^{T} y_t y_{t-1}'\right) \\ \widehat{\Sigma} &= \frac{1}{T-k}\sum_{t=k+1}^{T}\hat{u}_t(\hat{u}_t)' \\ \hat{u}_t &= y_t - \widehat{A}y_{t-1}\end{aligned}$$

Observações

Se o VAR for estacionário, então, os estimadores de MQO serão:

1. Consistentes;
2. Assintoticamente eficientes;
3. Se $u_t \sim N(0, \Sigma)$, os estimadores e as estatísticas de teste têm as distribuições assintóticas usuais.

Diagnóstico

Para testar se o modelo é adequado, usamos os resíduos (que guardam covariâncias contemporâneas) para construir a versão multivariada da estatística de Box-Ljung-Pierce, dada por

$$Q(m) = T^2 \sum_{\tau=1}^{m} \frac{1}{T-\tau} \text{tr}\left\{\widehat{\Gamma}(\tau)'\widehat{\Gamma}(0)^{-1}\widehat{\Gamma}(\tau)\widehat{\Gamma}(0)^{-1}\right\}$$

em que, H_0: não existe correlação serial no vetor de erros até a m-ésima defasagem.

Sob H_0, $Q(m) \sim \chi^2_{q^2(m-p)}$.

Outros testes

Teste LM para autocorrelação

H0: não existe correlação serial no vetor de erros na m-ésima defasagem

Teste de Normalidade

Jarque-Bera Multivariado

Structural VAR

Rubio-Ramírez, Waggoner and Zha (2010) say that

> *"Since the seminal work by Sims (1980), identification of structural vector autoregressions (SVARs) has been an unresolved theoretical issue.*
>
> *Filling this theoretical gap is of vital importance because impulse responses based on SVARs have been widely used for policy analysis and to provide stylized facts for dynamic stochastic general equilibrium (DSGE) models."*

Example

Let

- $P_{c,t}$ is the price index of commodities.
- Y_t is output.
- R_t is the nominal short-term interest rate.

Trivariate SVAR(1) representation:

$$
\begin{aligned}
a_{11}\Delta \log P_{c,t} + 0.0 \log Y_t + a_{31} R_t &= c_1 + b_{11}\Delta \log P_{c,t-1} + b_{21}\Delta \log Y_{t-1} + b_{31} R_{t-1} + \varepsilon_{1,t} \\
a_{12}\Delta \log P_{c,t} + a_{22} \log Y_t + 0.0 R_t &= c_2 + b_{12}\Delta \log P_{c,t-1} + b_{22}\Delta \log Y_{t-1} + b_{32} R_{t-1} + \varepsilon_{2,t} \\
a_{13}\Delta \log P_{c,t} + a_{23} \log Y_t + a_{33} R_t &= c_3 + b_{13}\Delta \log P_{c,t-1} + b_{23}\Delta \log Y_{t-1} + b_{33} R_{t-1} + \varepsilon_{3,t}
\end{aligned}
$$

1st eq. monetary policy equation.
2nd eq. characterizes behaviour of finished-goods producers.
3rd eq. commodity prices are set in active competitive markets.

Model set up

The (basic) SVAR(p) can be written as

$$A_0 y_t = A_1 y_{t-1} + \ldots + A_p y_{t-p} + u_t \qquad u_t \sim \text{i.i.d. } N(0, I_q),$$

where

- $A = (A_1, \ldots, A_p)$
- $B_i = A_0^{-1} A_i \quad i = 1, \ldots, p$
- $B = A_0^{-1} A$
- $\Sigma = (A_0 A_0')^{-1}$

Much of the SVAR literature involves exactly identified models.

Exact Identification

Define g such that $g(A_0, A) = (A_0^{-1}A, (A_0A_0')^{-1})$.

Consider an SVAR with restrictions represented by R.

Definition: The SVAR is exactly identified if and only if, for almost any reduced-form parameter point (B, Σ), there exists a unique structural parameter point $(A_0, A) \in R$ such that $g(A_0, A) = (B, \Sigma)$.

Waggoner and Zha (2003) developed an efficient MCMC algorithm to generate draws from a restricted A_0 matrix.

Illustration 1

$$A_0 = \begin{pmatrix} & PS & PS & MP & MD & Inf \\ \log Y & a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ \log P & 0 & a_{22} & 0 & a_{14} & a_{25} \\ R & 0 & 0 & a_{33} & a_{34} & a_{35} \\ \log M & 0 & 0 & a_{43} & a_{44} & a_{45} \\ \log P_c & 0 & 0 & 0 & 0 & a_{15} \end{pmatrix}$$

where

- $\log Y$: log gross domestic product (GDP)
- $\log P$: log GDP deflator
- R: nominal short-term interest rate
- $\log M$: log M3
- $\log P_c$: log commodity prices

and

- MP: monetary policy (central bank's contemporaneous behavior)
- Inf: commodity (information) market
- MD: money demand equation
- PS: production sector

Illustration 2

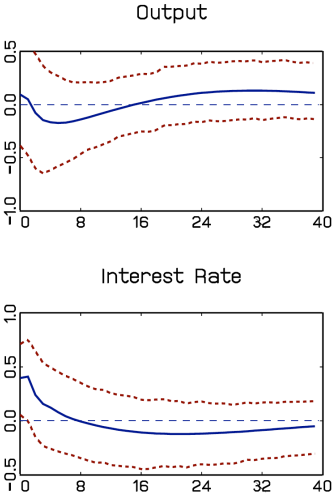
$$A_0 = \begin{pmatrix} & PCOM & M2 & R & Y & CPI & U \\ Inform & X & X & X & X & X & X \\ MP & 0 & X & X & 0 & 0 & 0 \\ MD & 0 & X & X & X & X & 0 \\ Prod & 0 & 0 & 0 & X & 0 & 0 \\ Prod & 0 & 0 & 0 & X & X & 0 \\ Prod & 0 & 0 & 0 & X & X & X \end{pmatrix}$$

where

- PCOM: Price index for industrial commodities
- M2: Real money
- R: Federal funds rate (R)
- Y: real GDP interpolated to monthly frequency
- CPI: Consumer price index (CPI)
- U: Unemployment rate (U)

- Inform: Information market
- MP: Monetary policy rule
- MD: Money demand
- Prod: Production sector of the economy

Monetary Policy Shock



VAR-GARCH

Pelloni and Polasek (2003) introduce the VAR model with GARCH errors as

$$y_t = \sum_{i=1}^{p} B_i y_{t-i} + u_t$$

where

$$u_t \sim N(0, \Sigma_t)$$

and

$$\text{vech}(\Sigma_t) = \alpha_0 + \sum_{i=1}^{r} A_i \text{vech}(\Sigma_{t-i}) + \sum_{i=1}^{s} \Theta_i \text{vech}(u_{t-i} u'_{t-i})$$

Example

German, U.S., and U.K. quarterly data sets over the period 1968-1998. Variables are logs of aggregate employment and of the employment shares of the manufacturing, finance, trade, and construction sectors for U.S. and U.K.

Table IVa. Bayes factors for model selection using posterior log-marginal likelihoods.

$\log BF_{21}$	Country		
	Germany	U.K.	U.S.
VAR	239.04	353.67	100.09
EC-VARCH	3.15	2.83	4.39
CEC-VARCH	11.79	6.71	4.79
COIN-VARCH 1	2.41	9.70	2.36
COIN-VARCH 2	6.76	1.31	10.11

VAR-SV

Uhlig (1997) introduced stochastic volatility (SV) for the error term in BVARs:

$$y_t = \sum_{i=1}^{p} B_i y_{t-i} + u_t,$$

where

$$u_t \sim N(0, \Sigma_t) \quad \text{and} \quad \Sigma_t^{-1} = L_t L_t',$$

and dynamics

$$\begin{aligned} \Sigma_{t+1}^{-1} &= \frac{L_t \Theta_t L_t'}{\lambda} \\ \Theta_t &\sim B_q\left(\frac{\nu + pq}{2}, \frac{1}{2}\right). \end{aligned}$$

TVP-VAR-SV

Primiceri (2005) discusses VARs with time varying coefficients and stochastic volatility

$$y_t = \sum_{i=1}^{p} B_{it} y_{t-i} + u_t \qquad u_t \sim N(0, \Sigma_t)$$

with

$$\Sigma_t = (A_t)^{-1} D_t (A_t')^{-1},$$

and

$$A_t = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_{21,t} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \alpha_{q1,t} & \cdots & \alpha_{q,q-1,t} & 1 \end{pmatrix} \quad D_t = \begin{pmatrix} \sigma^2_{1,t} & 0 & \cdots & 0 \\ 0 & \sigma^2_{2,t} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma^2_{q,t} \end{pmatrix}.$$

Dynamics

VAR coefficients:

$$B_t = B_{t-1} + \nu_t \qquad \nu_t \sim N(0, Q)$$

Cholesky coefficients:

$$\alpha_t = \alpha_{t-1} + \xi_t \qquad \xi_t \sim N(0, S)$$

Stochastic volatility:

$$\log \sigma_t = \log \sigma_{t-1} + \eta_t \qquad \eta_t \sim N(0, W)$$

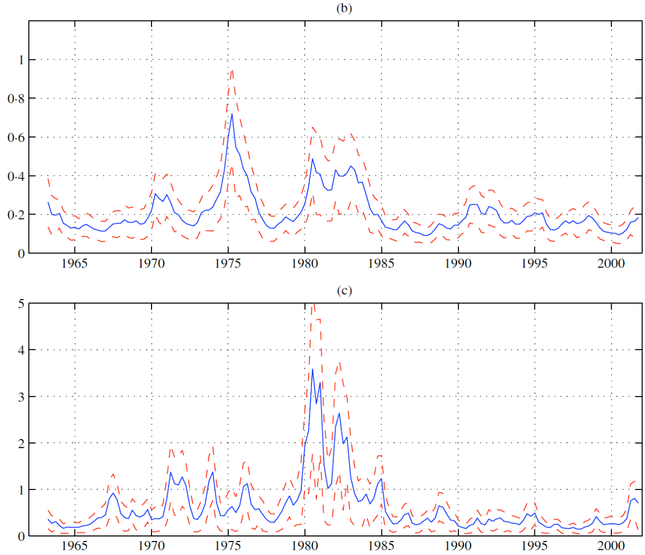


FIGURE 1

Posterior mean, 16-th and 84-th percentiles of the standard deviation of (a) residuals of the inflation equation, (b) residuals of the unemployment equation and (c) residuals of the interest rate equation or monetary policy shocks

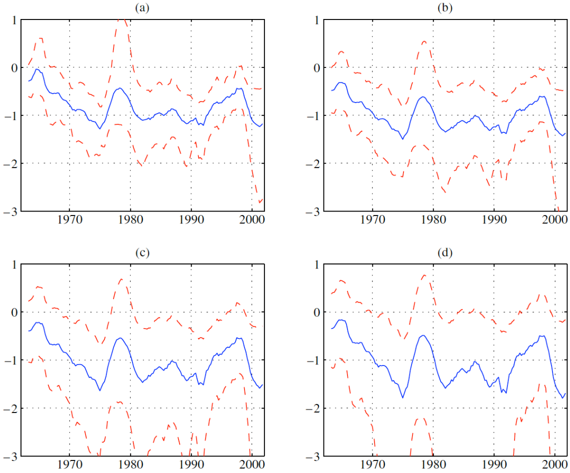


FIGURE 6

Interest rate response to a 1% permanent increase of unemployment with 16-th and 84-th percentiles. (a) Simultaneous response, (b) response after 10 quarters, (c) response after 20 quarters, (d) response after 60 quarters

See Nakajima, Kasuya and Watanabe (2011) for an application to the Japanese economy.

Curse of dimensionality

VAR(1) case[1]

Small: $q = 3 \;\Rightarrow$ 15 parameters

Medium: $q = 20 \;\Rightarrow$ 610 parameters

Large: $q = 131 \;\Rightarrow$ 25,807 parameters

See also Korobilis (2008) and Koop and Korobilis (2013) papers on forecasting in vector autoregressions with many predictors and large time-varying parameter VARs, respectively.

[1] Small, Medium and Large are based on the VAR specifications of Banbura, Giannone and Reichlin (2010).

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