Reduced form VAR In the United States of Wonderland the growth rates of income (GNP) and the money stock (M2) as well as an interest rate (IR) are related as in the following VAR(2) model

$$y_t = \nu + A_1 y_{t-1} + A_2 y_{t-2} + u_t$$

or, more explicitly,

$$\begin{pmatrix} \text{GNP}_t \\ \text{M2}_t \\ \text{IR}_t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.7 & 0.1 & 0.0 \\ 0.0 & 0.4 & 0.1 \\ 0.9 & 0.0 & 0.8 \end{pmatrix} \begin{pmatrix} \text{GNP}_{t-1} \\ \text{M2}_{t-1} \\ \text{IR}_{t-1} \end{pmatrix} + \begin{pmatrix} -0.2 & 0.0 & 0.0 \\ 0.0 & 0.1 & 0.1 \\ 0.0 & 0.0 & 0.0 \end{pmatrix} \begin{pmatrix} \text{GNP}_{t-2} \\ \text{M2}_{t-2} \\ \text{IR}_{t-2} \end{pmatrix} + u_t$$

where $u_t \sim (0, \Sigma)$ and

$$\Sigma = \begin{pmatrix} 0.26 & 0.03 & 0.00 \\ 0.03 & 0.09 & 0.00 \\ 0.00 & 0.00 & 0.81 \end{pmatrix}.$$

We have shown that this is a stable VAR(2). Therefore, its VMA(∞) representation is

$$y_t = \mu + \sum_{i=0}^{\infty} \Phi_i u_{t-i},$$

where $\mu = (I - A_1 - A_2)^{-1}\nu = (6.88, 14.38, 30.94)'$, $\Phi_0 = I_q$, $\Phi_1 = A_1$ and $\Phi_i = \Phi_{i-1}A_1 + \Phi_{i-2}A_2$, for $i \geq 2$. For example,

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} 0.7 & 0.1 & 0.0 \\ 0.0 & 0.4 & 0.1 \\ 0.9 & 0.0 & 0.8 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 0.29 & 0.11 & 0.01 \\ 0.09 & 0.26 & 0.22 \\ 1.35 & 0.09 & 0.64 \end{pmatrix}.$$

Figure 1 plots entries (i, j) of Φ_hs for $h = 0, \ldots, 50$.

Orthogonal shocks. It is desirable to study the behavior of y_t to non correlated (orthogonal) shocks, say ϵ_t, instead of correlated (non-orthogonal) shocks u_t as above. Therefore, we need to *orthogonalize* the shocks u_t and two most commonly known automatic methods to do so are based on the *Cholesky decomposition* and the *singular value decomposition* of Σ.

- *Cholesky decomposition:* Σ is rewritten as

$$\Sigma = LDL',$$

where L is lower-triangular with ones in the diagonal and D is a diagonal matrix with positive values in the main diagonal. It follows immediately that

$$D = L^{-1}\Sigma(L^{-1})'.$$

- *Singular value decomposition:* Σ is rewritten as

$$\Sigma = U\Lambda U',$$

where the columns of U are the orthogonal (ie. $U' = U^{-1}$) eigenvectors of Σ and Λ is a diagonal matrix with positive values in the main diagonal representing the eigenvalues of Σ. It follows immediately that

$$\Lambda = U'\Sigma U.$$

We now have a strategy to transform u_t into orthogonal shocks. Let

$$\begin{aligned} \epsilon_t &= L^{-1}u_t \\ \eta_t &= U'u_t. \end{aligned}$$

Then

$$\begin{aligned} E(\epsilon_t) &= E(L^{-1}u_t) = L^{-1}E(u_t) = 0 \\ V(\epsilon_t) &= V(L^{-1}u_t) = L^{-1}V(u_t)(L^{-1})' = L^{-1}\Sigma(L^{-1})' = D, \end{aligned}$$

and

$$\begin{aligned} E(\eta_t) &= E(U'u_t) = U'E(u_t) = 0 \\ V(\eta_t) &= V(U'u_t) = U'V(u_t)U = U'\Sigma U = \Lambda. \end{aligned}$$

The new orthogonal VMA(∞) representations are

$$\begin{aligned} y_t &= \mu + \sum_{i=0}^{\infty} \Phi_i L L^{-1} u_t = \mu + \sum_{i=0}^{\infty} \tilde{\Phi}_i \epsilon_t \\ y_t &= \mu + \sum_{i=0}^{\infty} \Phi_i L U u_t = \mu + \sum_{i=0}^{\infty} \Phi_i^* \eta_t, \end{aligned}$$

where $\tilde{\Phi}_i = \Phi_i L$ and $\Phi_i^* = \Phi_i U$ for $i = 0, 1, \ldots$. Finally, it worth mentioning that the shocks u_t, ϵ_t and η_t have completely different interpretations.

In our example, (D, L) and (Λ, U) are

$$D = \begin{pmatrix} 0.519 & 0.0 & 0.0 \\ 0.0 & 0.294 & 0.0 \\ 0.0 & 0.0 & 0.9 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.12 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} 0.81 & 0.0 & 0.0 \\ 0.0 & 0.265 & 0.0 \\ 0.0 & 0.0 & 0.085 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0.0 & -1.00 & -0.17 \\ 0.0 & -0.17 & 1.00 \\ 1.0 & 0.0 & 0.0 \end{pmatrix}$$

respectively. Figures 2 and 3 plot entries (i, j) of $\tilde{\Phi}_h$s and Φ_h^*s for $h = 0, \ldots, 50$.

Structural VAR. The transformations of shocks u_t into orthogonal shocks ϵ_t and η_t via

$$\begin{aligned} \epsilon_t &= L^{-1}u_t \\ \eta_t &= U'u_t, \end{aligned}$$

can also be used to transform the reduced form VAR(2):

$$y_t = \nu + A_1 y_{t-1} + A_2 y_{t-2} + u_t$$

into two structural form VAR(2):

$$\begin{aligned} L^{-1}y_t &= L^{-1}\nu + L^{-1}A_1 y_{t-1} + L^{-1}A_2 y_{t-2} + L^{-1}u_t \\ U'y_t &= U'\nu + U'A_1 y_{t-1} + U'A_2 y_{t-2} + U'u_t \end{aligned}$$

or simply

$$\begin{aligned} L^{-1}y_t &= \tilde{\nu} + \tilde{A}_1 y_{t-1} + \tilde{A}_2 y_{t-2} + \epsilon_t \\ U'y_t &= \nu^* + A_1^* y_{t-1} + A_2^* y_{t-2} + \eta_t. \end{aligned}$$

Therefore,

$$\begin{aligned} L^{-1}y_t &= \begin{pmatrix} 1.00 & 0.00 & 0.00 \\ -0.12 & 1.0 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{pmatrix} \begin{pmatrix} \text{GNP}_t \\ \text{M2}_t \\ \text{IR}_t \end{pmatrix} = \begin{pmatrix} \text{GNP}_t \\ \text{M2}_t - 0.12\text{GNP}_t \\ \text{IR}_t \end{pmatrix} \\ &= \begin{pmatrix} 2.00 \\ 0.77 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 0.70 & 0.10 & 0.0 \\ -0.08 & 0.39 & 0.1 \\ 0.90 & 0.00 & 0.8 \end{pmatrix} \begin{pmatrix} \text{GNP}_{t-1} \\ \text{M2}_{t-1} \\ \text{IR}_{t-1} \end{pmatrix} + \begin{pmatrix} -0.20 & 0.0 & 0.0 \\ 0.02 & 0.1 & 0.1 \\ 0.00 & 0.0 & 0.0 \end{pmatrix} \begin{pmatrix} \text{GNP}_{t-2} \\ \text{M2}_{t-2} \\ \text{IR}_{t-2} \end{pmatrix} + \epsilon_t. \end{aligned}$$

Similarly,

$$\begin{aligned} U'y_t &= \begin{pmatrix} 0.00 & 0.00 & 1.00 \\ -1.00 & -0.17 & 0.00 \\ -0.17 & 1.00 & 0.00 \end{pmatrix} \begin{pmatrix} \text{GNP}_t \\ \text{M2}_t \\ \text{IR}_t \end{pmatrix} = \begin{pmatrix} \text{IR}_t \\ -\text{GNP}_t - 0.17\text{M2}_t \\ -0.17\text{GNP}_t + \text{M2}_t \end{pmatrix} \\ &= \begin{pmatrix} 0.00 \\ -2.14 \\ 0.65 \end{pmatrix} + \begin{pmatrix} 0.90 & 0.00 & 0.80 \\ -0.69 & -0.17 & -0.02 \\ -0.12 & 0.38 & 0.10 \end{pmatrix} \begin{pmatrix} \text{GNP}_{t-1} \\ \text{M2}_{t-1} \\ \text{IR}_{t-1} \end{pmatrix} + \begin{pmatrix} 0.00 & 0.00 & 0.00 \\ 0.20 & -0.02 & -0.02 \\ 0.03 & 0.10 & 0.10 \end{pmatrix} \begin{pmatrix} \text{GNP}_{t-2} \\ \text{M2}_{t-2} \\ \text{IR}_{t-2} \end{pmatrix} + \eta_t \end{aligned}$$

As we mentioned before, the shocks ϵ_t and η_t have completely different interpretations, which applies directly to the structural VAR based on them.

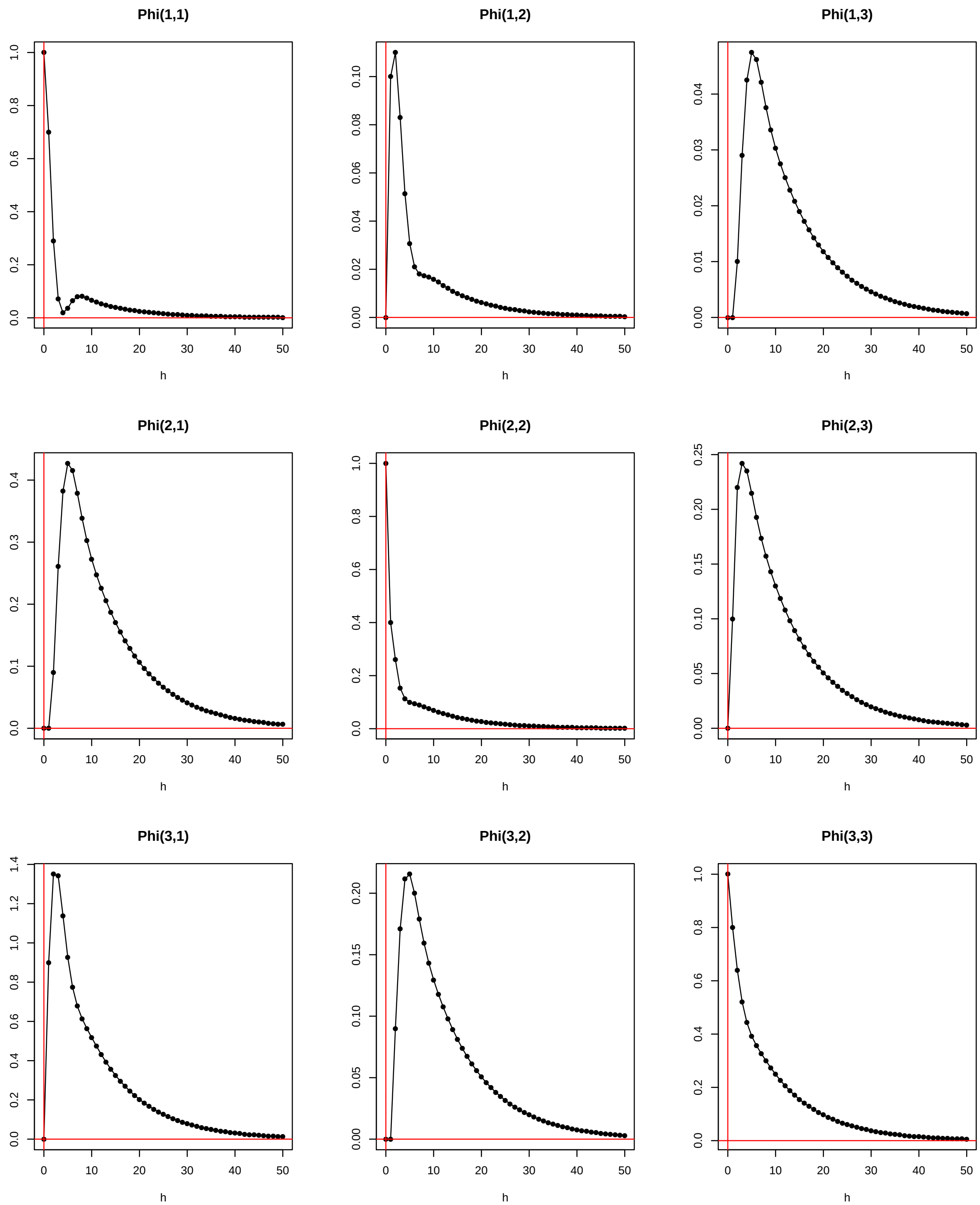

Figure 1: Reduced form VMA representation matrices $\Phi_0, \Phi_1, \ldots, \Phi_{50}$.

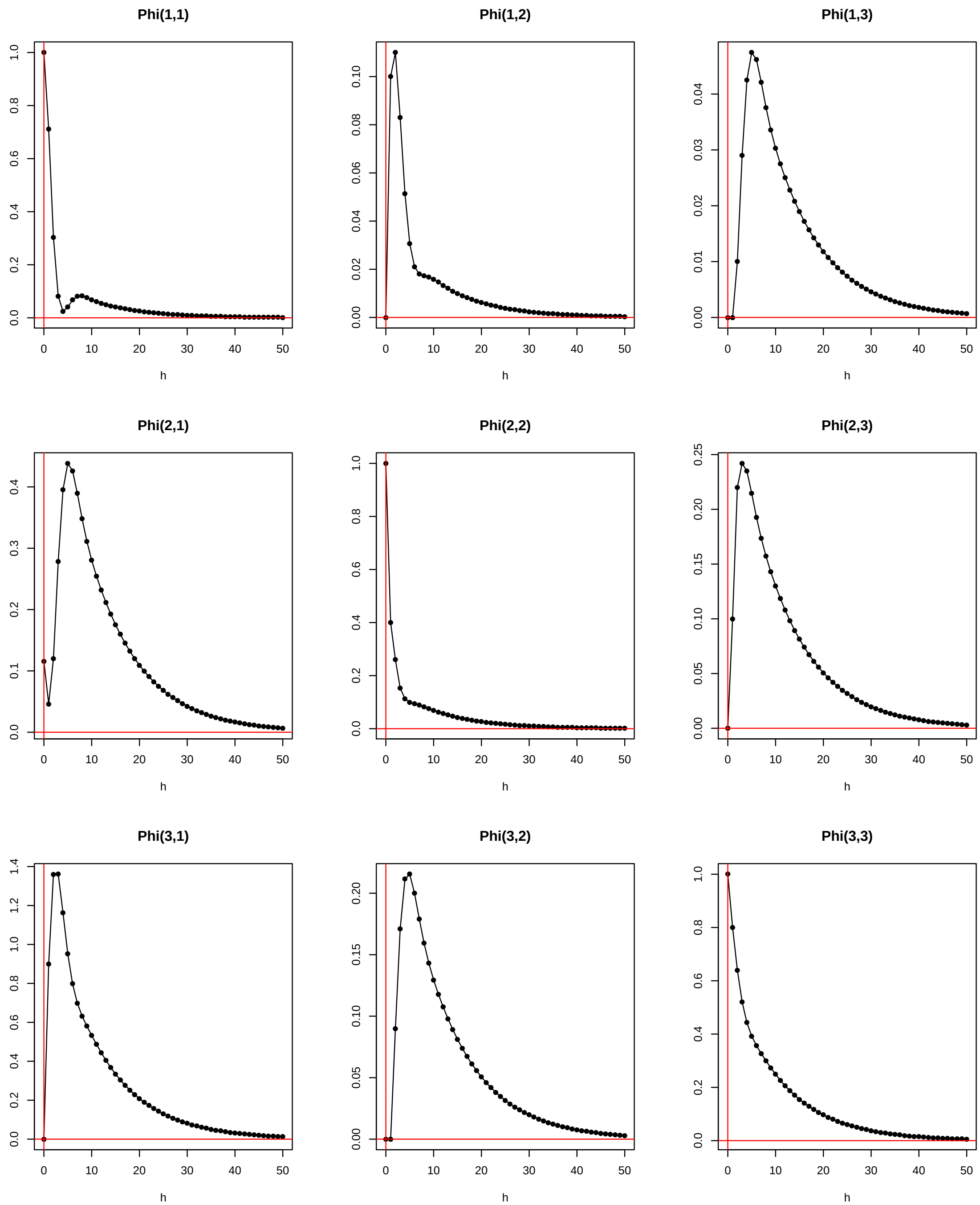

Figure 2: Cholesky-based structural form VMA representation matrices $\tilde{\Phi}_0, \tilde{\Phi}_1, \ldots, \tilde{\Phi}_{50}$.

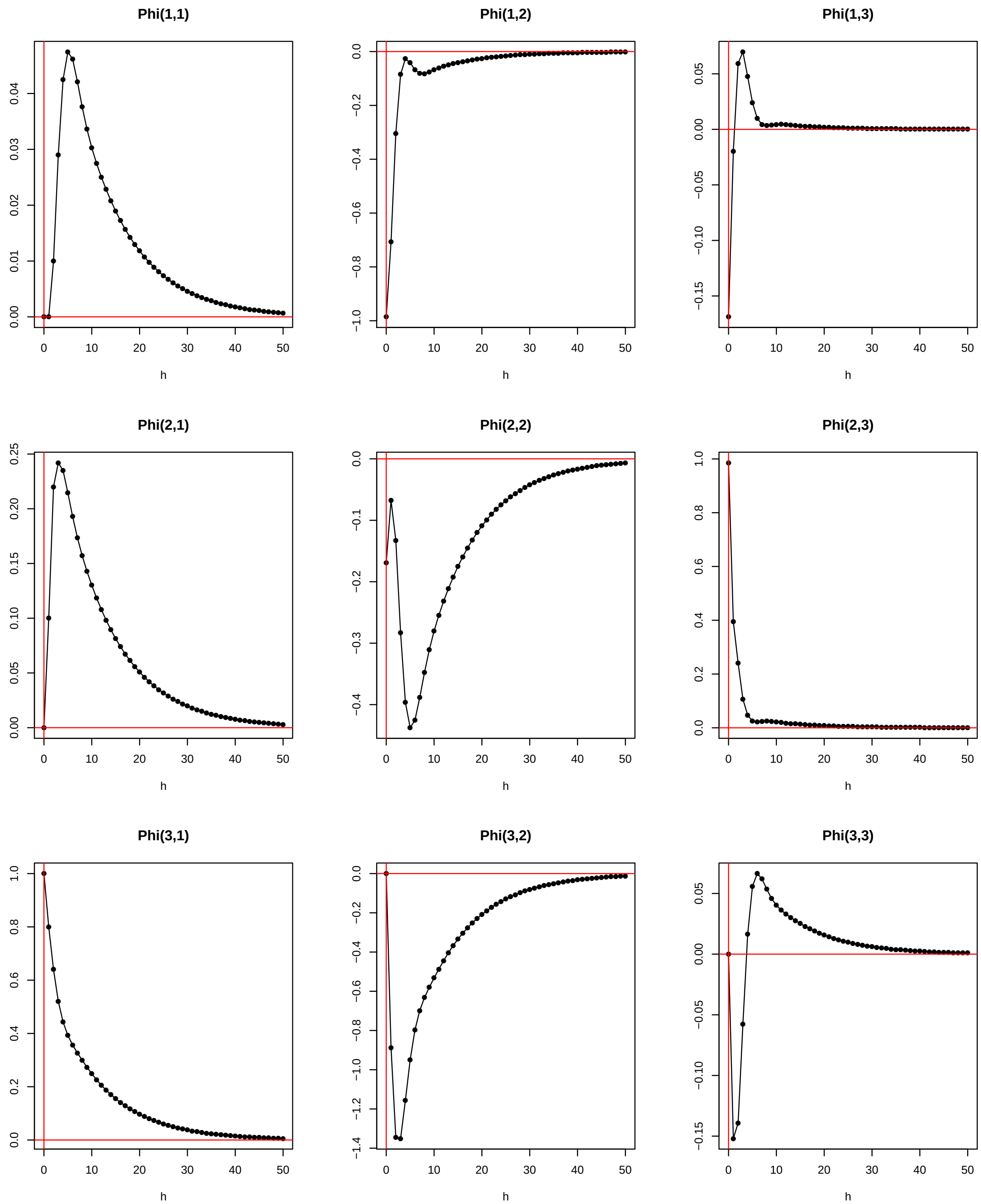

Figure 3: Singular-value-decomposition-based structural form VMA representation matrices $\Phi_0^*, \Phi_1^*, \ldots, \Phi_{50}^*$.