Stochastic Volatility (SV) Models
Lecture 9

Morettin & Toloi, 2006, Section 14.6
Tsay, 2010, Section 3.12
Tsay, 2013, Section 4.13

Stochastic volatility model

The canonical stochastic volatility model (SV-AR(1), hereafter), is a state-space model where the state variable is the log-volatility:

$$\begin{aligned} r_t|h_t &\sim N(0, \exp\{h_t\}) \\ h_t|h_{t-1} &\sim N(\mu + \phi(h_{t-1} - \mu), \sigma^2) \end{aligned}$$

where $\mu \in \Re$, $|\beta| < 1$, $\sigma^2 > 0$ and $h_0 \sim N(\mu, \sigma^2/(1 - \phi^2))$.

μ: unconditional mean of log-volatility.

$\sigma^2/(1 - \phi^2)$: unconditional variance of log-volatility.

σ^2: conditional variance of log-volatility.

$|\beta| < 1$: log-volatility follows a stationary process.

Nonlinear dynamic model

Noticing that $r_t | h_t \sim N(0, \exp\{h_t\})$ is equivalent to

$$r_t = \exp\{h_t/2\}\varepsilon_t.$$

the model can be rewritten as

$$\begin{aligned} \log r_t^2 &= h_t + \log \varepsilon_t^2 \\ h_t &= \alpha + \phi h_{t-1} + \sigma \eta_t \end{aligned}$$

which looks like a standard dynamic linear model, for $\alpha = \mu(1 - \phi)$.

Observational error, $\log \varepsilon_t^2$, is no longer Gaussian!

In fact, $\log \varepsilon_t^2 \sim \log \chi_1^2$, where

$$\begin{aligned} E(\log \varepsilon_t^2) &= -1.27 \\ V(\log \varepsilon_t^2) &= \frac{\pi^2}{2} = 4.935 \end{aligned}$$

Normal approximation

Let $z_t = \log r_t^2 + 1.27$ and $\omega^2 = \pi^2/2$. Then, the normal DLM approximation to the SV-AR(1) model is:

$$\begin{aligned} z_t &= h_t + \omega v_t \\ h_t &= \alpha + \phi h_{t-1} + \sigma \eta_t, \end{aligned}$$

so the Kalman filter and smoother can then be easily implemented.

The parameters (μ, ϕ, τ) can estimated either via maximum likelihood or via Bayesian inference.

Main issue: Normal approximation is usually quite poor!

A bit of Bayesian inference

A commonly used prior set up is

- Prior for μ:

$$\mu \sim N(b_\mu, B_\mu^2)$$

- Prior for ϕ:

$$\frac{\phi + 1}{2} \sim Beta(a_0, b_0),$$

so that

$$E(\phi) = \frac{2a_0}{a_0 + b_0} - 1 \quad \text{and} \quad V(\phi) = \frac{4a_0 b_0}{(a_0 + b_0)^2(a_0 + b_0 + 1)}$$

- Prior for σ^2:

$$\sigma^2 \sim Gamma(1/2, 1/(2B_\sigma)),$$

so that $E(\sigma^2) = B_\sigma$.

R package stochvol

Efficient Bayesian Inference for SV Models: stochvol runs an MCMC scheme for burnin + draws iterations from the posterior distribution $p(h_1, \ldots, h_n, \mu, \phi, \sigma | y_1, \ldots, y_n)$.

```
svsample(y,draws=10000,burnin=1000,priormu=c(-10,3),
         priorphi=c(5,1.5),priorsigma=1,thinpara=1,
         thinlatent=1,thintime=1,quiet=FALSE,
         startpara,startlatent,expert,...)
```

where priormu$= (b_\mu, B_\mu)$, priorphi$= (a_0, b_0)$ and priorsigma$= B_\sigma$.

Only draws/thinpara draws are kept for (μ, ϕ, σ).

Only draws/thinlatent draws are kept for $(h_1, h_2, \ldots, h_n)$.

Kastner and Frühwirth-Schnatter (2013) Ancillarity-sufficiency interweaving strategy for boosting MCMC estimation of stochastic volatility models.

Example

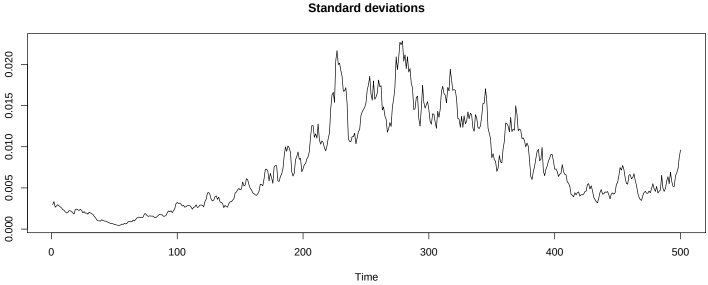
Simulating a time series with $n = 500$ observations:

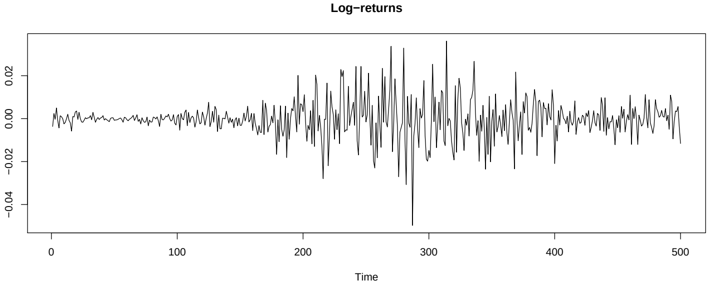
```
sim   = svsim(500,mu=-10,phi=0.99,sigma=0.2)
```

Running the MCMC scheme:

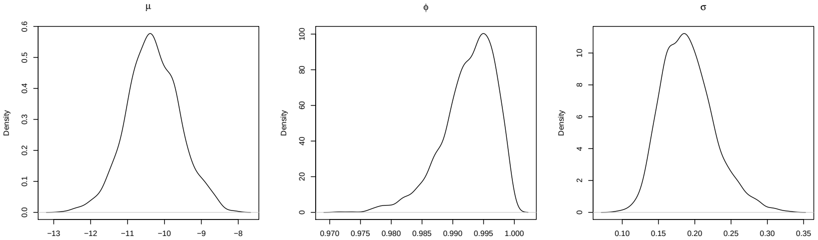
```
draws = svsample(sim$y,draws=200000,burnin=1000,
                 thinpara=100,thinlatent=100,
                 priormu=c(-10,1),
                 priorphi=c(20,1.2),
                 priorsigma=0.2)
```

sim=svsim(500,mu=-10,phi=0.99,sigma=0.2)

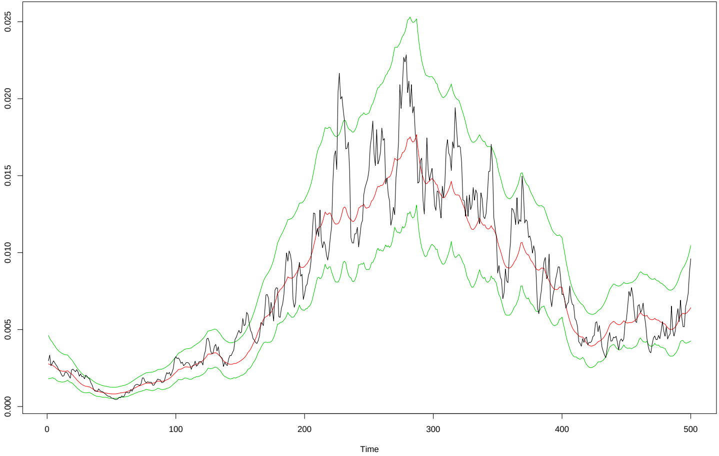




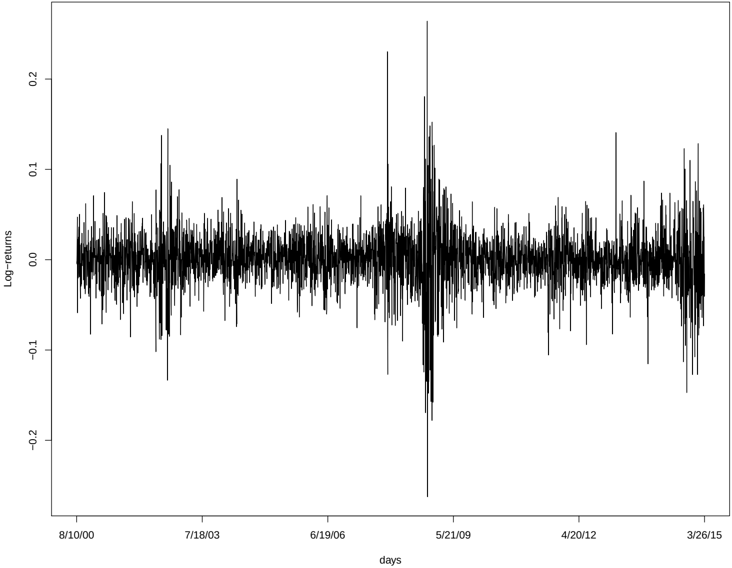
Posterior of μ, ϕ and σ



Standard deviations

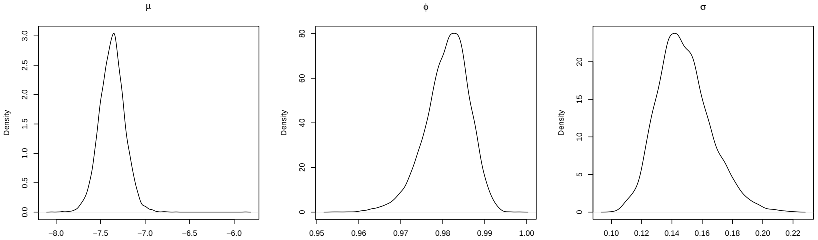


Petrobrás

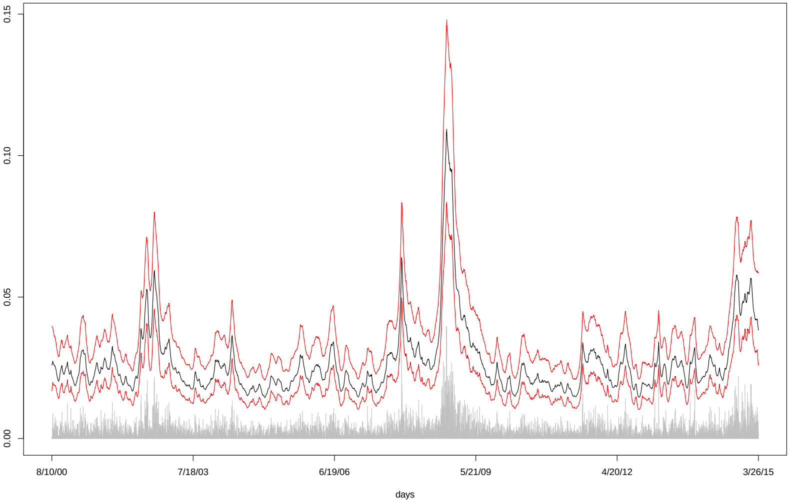


0.2
0.1
0.0
−0.1
−0.2
Log−returns
8/10/00
7/18/03
6/19/06
5/21/09
4/20/12
3/26/15
days

Posterior of μ, ϕ and σ



Standard deviations



GARCH(1,1) vs SV-AR(1)

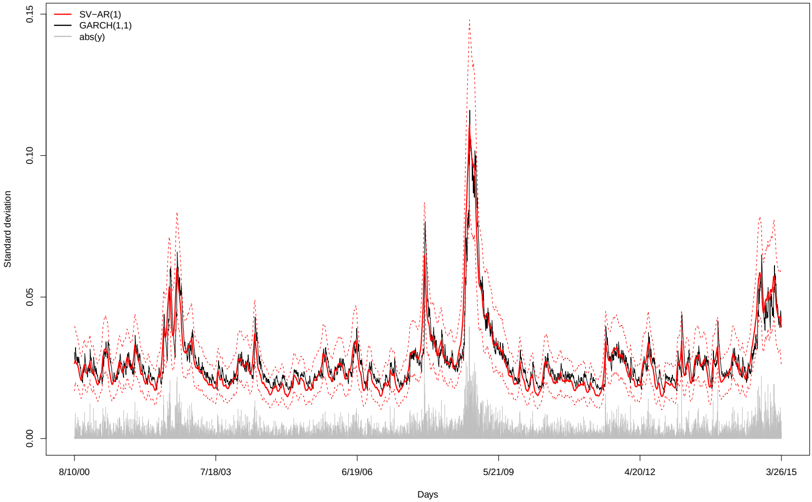
MAXIMUM LIKELIHOOD ESTIMATION

	Estimate	Std. Error
omega	0.00001473	0.00000315
alpha1	0.07196618	0.00804195
beta1	0.91000140	0.01009689

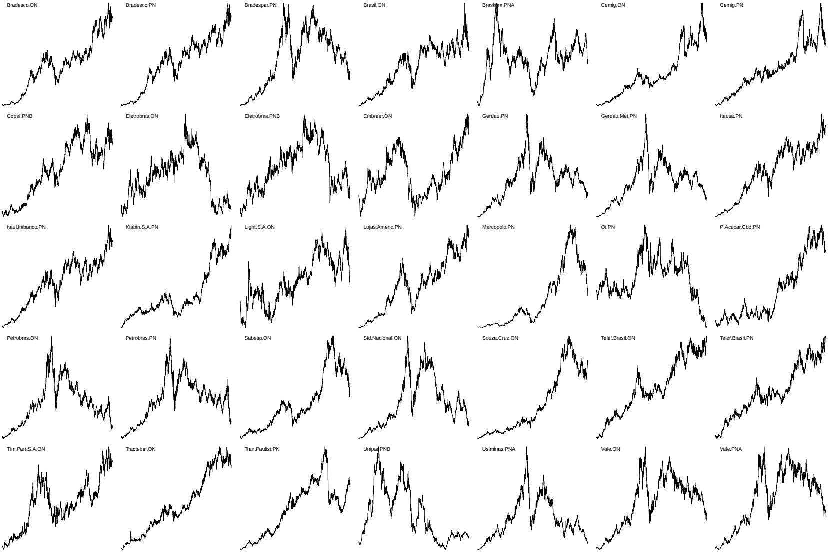
BAYESIAN ESTIMATION

	mu	phi	sigma
1st Qu.	-7.464	0.9779	0.1359
Median	-7.371	0.9816	0.1464
Mean	-7.371	0.9811	0.1480
3rd Qu.	-7.282	0.9847	0.1585

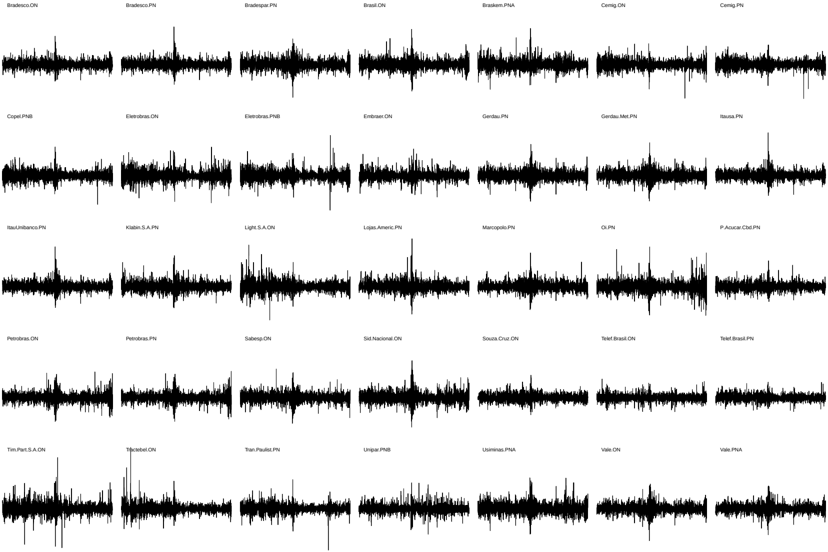
GARCH(1,1) vs SV-AR(1)



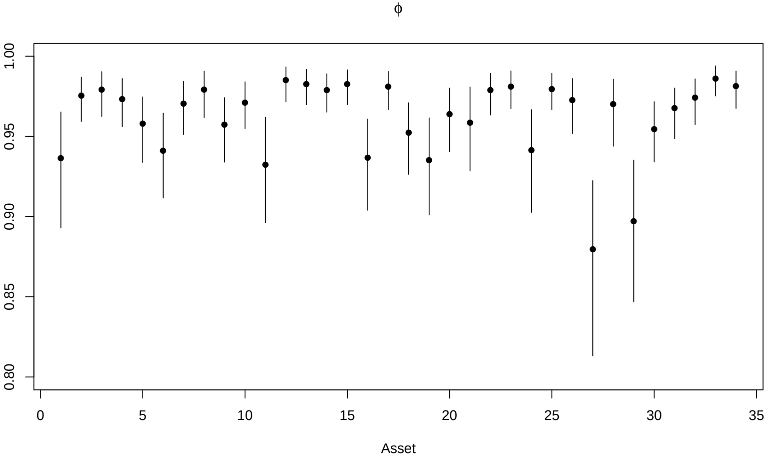
Brazilian market: Jan 2nd 2003 - Feb 9th 2015



Returns


Bradesco.ON
Bradesco.PN
Bradespar.PN
Brasil.ON
Braskem.PNA
Cemig.ON
Cemig.PN
Copel.PNB
Eletrobras.ON
Eletrobras.PNB
Embraer.ON
Gerdau.PN
Gerdau.Met.PN
Itausa.PN
ItauUnibanco.PN
Klabin.S.A.PN
Light.S.A.ON
Lojas.Americ.PN
Marcopolo.PN
Oi.PN
P.Acucar.Cbd.PN
Petrobras.ON
Petrobras.PN
Sabesp.ON
Sid.Nacional.ON
Souza.Cruz.ON
Telef.Brasil.ON
Telef.Brasil.PN
Tim.Part.S.A.ON
Tractebel.ON
Tran.Paulist.PN
Unipar.PNB
Usiminas.PNA
Vale.ON
Vale.PNA

Volatility persistance



Lopes and Salazar (2006)[1]

We extend the SV-AR(1) where

$$y_t \sim N(0, \exp\{h_t\})$$

to accommodate a smooth regime shift, i.e.

$$h_t \sim N(\alpha_{1t} + F(\gamma, \kappa, h_{t-d})\alpha_{2t}, \sigma^2)$$

where

$$\begin{aligned} \alpha_{it} &= \mu_i + \phi_i h_{t-1} + \delta_i h_{t-2} \qquad i = 1, 2 \\ F(\gamma, \kappa, h_{t-d}) &= \frac{1}{1 + exp\left(\gamma(\kappa - h_{t-d})\right)} \end{aligned}$$

such that $\gamma > 0$ drives smoothness and c is a threshold.

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[1] Time series mean level and stochastic volatility modeling by smooth transition autoregressions: a Bayesian approach, In Fomby, T.B. (Ed.) *Advances in Econometrics: Econometric Analysis of Financial and Economic Time Series/Part B*, Volume 20, 229-242.

Modeling S&P500 returns

Data from Jan 7th, 1986 to Dec 31st, 1997 (3127 observations)

Models	AIC	BIC	DIC
AR(1)	12795	31697	7223.1
AR(2)	12624	31532	7149.2
LSTAR(1,d=1)	**12240**	**31165**	**7101.1**
LSTAR(1,d=2)	12244	31170	7150.3
LSTAR(2,d=1)	12569	31507	7102.4
LSTAR(2,d=2)	12732	31670	7159.4

Modeling S&P500 returns

Parameter	Models					
	AR(1)	*AR*(2)	LSTAR(1,1)	LSTAR(1,1)	LSTAR(2,1)	LSTAR(2,1)
	Posterior mean (standard deviation)					
μ_1	-0.060	-0.066	0.292	-0.354	-4.842	-6.081
	(0.184)	(0.241)	(0.579)	(0.126)	(0.802)	(1.282)
ϕ_1	0.904	0.184	0.306	0.572	-0.713	-0.940
	(0.185)	(0.242)	(0.263)	(0.135)	(0.306)	(0.699)
δ_1	-	0.715	-	-	-1.018	-1.099
		(0.248)			(0.118)	(0.336)
μ_2	-	-	-0.685	0.133	4.783	6.036
			(0.593)	(0.092)	(0.801)	(1.283)
ϕ_2	-	-	0.794	0.237	0.913	1.091
			(0.257)	(0.086)	(0.314)	(0.706)
δ_2	-	-	-	-	1.748	1.892
					(0.114)	(0.356)
γ	-	-	118.18	163.54	132.60	189.51
			(16.924)	(23.912)	(10.147)	(0.000)
κ	-	-	-1.589	0.022	-2.060	-2.125
			(0.022)	(0.280)	(0.046)	(0.000)
σ^2	0.135	0.234	0.316	0.552	0.214	0.166
	(0.020)	(0.044)	(0.066)	(0.218)	(0.035)	(0.026)

Carvalho and Lopes (2007)[2]

We extend the SV-AR(1) to accommodate a Markovian regime shift, i.e.

$$h_t \sim N(\mu_{s_t} + \phi h_{t-1}, \sigma^2)$$

and

$$Pr(s_t = j | s_{t-1} = i) = p_{ij} \qquad \text{for } i, j = 1, \ldots, k.x$$

for

$$\alpha_{s_t} = \gamma_1 + \sum_{j=1}^{k} \gamma_j I_{jt}$$

where $I_{jt} = 1$ if $s_t \geq j$ and zero otherwise, $\gamma_1 \in \Re$ and $\gamma_i > 0$ for $i > 1$.

[2] Simulation-based sequential analysis of Markov switching stochastic volatility models, *Computational Statistics and Data Analysis*, 51, 4526-4542.

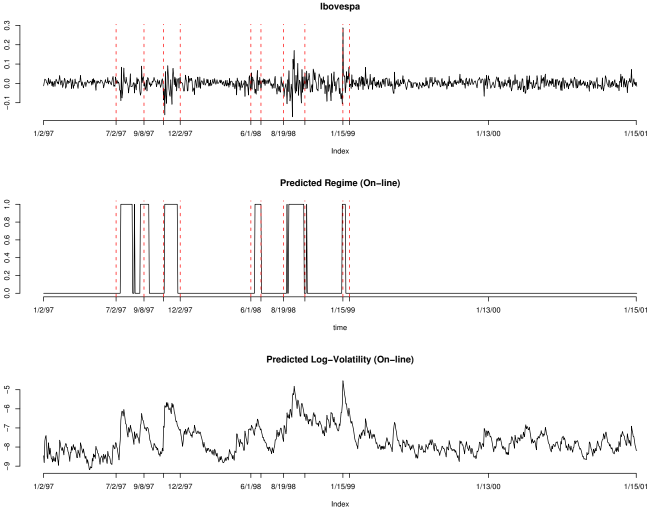
Modeling IBOVESPA returns

We analyzed IBOVESPA returns from 01/02/1997 to 01/16/2001 (1000 observations) based on a 2-regime model.

07/02/1997	Thailand devalues the baht by as much as 20%.
08/11/1997	IMF and Thailand set a rescue agreement.
10/23/1997	Hong Kong's stock index falls 10.4%. South Korea Won starts to weaken.
12/02/1997	IMF and South Korea set a bailout agreement.
06/01/1998	Russia's stock market crashes.
06/20/1998	IMF gives final approval to a loan package to Russia.
08/19/1998	Russia officially falls into default.
10/09/1998	IMF and World Bank joint meeting to discuss the global economic crisis. The Fed cuts interest rates.
01/15/1999	The Brazilian government allows its currency, the real, to float freely by lifting exchange controls.
02/02/1999	Arminio Fraga is named president of Brazil's Central Bank.

Model	95% credible interval	$E(\phi \mid D_T)$
SV	(0.9325;0.9873)	0.9525
MSSV	(0.8481;0.8903)	0.8707

Also, $E(p_{11} \mid D_T) = 0.993$ and $E(p_{11} \mid D_T) = 0.964$.



Ibovespa
-0.1 0.0 0.1 0.2 0.3
1/2/97 7/2/97 9/8/97 12/2/97 6/1/98 8/19/98 1/15/99 1/13/00 1/15/01
Index
Predicted Regime (On-line)
0.0 0.2 0.4 0.6 0.8 1.0
1/2/97 7/2/97 9/8/97 12/2/97 6/1/98 8/19/98 1/15/99 1/13/00 1/15/01
time
Predicted Log-Volatility (On-line)
-9 -8 -7 -6 -5
1/2/97 7/2/97 9/8/97 12/2/97 6/1/98 8/19/98 1/15/99 1/13/00 1/15/01
Index

Abanto, Migon and Lopes (2009)[3]

We use a modied mixture model with Markov switching volatility specfication to analyze the relationship between stock return volatility and trading volume, i.e.

$$
\begin{array}{rcl}
y_t|h_t & \sim & t_\nu(0, \exp\{h_t\}) \\
v_t|h_t & \sim & Poisson(m_0 + m_1 \exp\{h_t\}) \\
h_t & \sim & N(\mu + \gamma s_t + \phi h_{t-1}, \tau^2)
\end{array}
$$

with $s_t = 0$ or $s_t = 1$, $\mu \in \Re$ and $\gamma < 0$.

[3] Bayesian modeling of financial returns: a relationship between volatility and trading volume. *Applied Stochastic Models in Business and Industry*, **26**, 172-193

Lopes and Polson (2010)[4]

The *stochastic volatility with correlated jumps* (SVCJ) model of Eraker, Johannes and Polson (2003) can be written as

$$
\begin{aligned}
y_{t+1} &= y_t + \mu\Delta + \sqrt{v_t\Delta}\epsilon^y_{t+1} + J^y_{t+1} \\
v_{t+1} &= v_t + \kappa(\theta - v_t) + \sigma_v\sqrt{v_t\Delta}\epsilon^v_{t+1} + J^v_{t+1}
\end{aligned}
$$

where both ϵ^y_{t+1} and ϵ^v_{t+1} follow $N(0,1)$ with $\text{corr}(\epsilon^y_{t+1}, \epsilon^v_{t+1}) = \rho$; and jump components

$$
\begin{aligned}
J^y_{t+1} &= \xi^y_{t+1} N_{t+1} \qquad J^v_{t+1} = \xi^v_{t+1} N_{t+1} \\
\xi^v_{t+1} &\sim Exp(\mu_v) \\
\xi^y_{t+1}|\xi^v_{t+1} &\sim N(\mu_y + \rho_J \xi^v_{t+1}, \sigma^2_y) \\
Pr(N_{t+1} = 1) &= \lambda\Delta
\end{aligned}
$$

Usually, $\Delta = 1$.

[4]Extracting SP500 and NASDAQ volatility: The credit crisis of 2007-2008. *Handbook of Applied Bayesian Analysis*.

Credit crisis of 2007

SV model: $\mu = J^y_{t+1} = J^v_{t+1} = 0$ and $\sqrt{v_t\Delta} = 1$ in the evolution equation.

SP500	Mean	StDev	2.5%	97.5%
$\kappa\theta$	-0.0031	0.0029	-0.0092	0.0022
$1-\kappa$	0.9949	0.0036	0.9868	1.0011
σ^2_v	0.0076	0.0026	0.0041	0.0144

SVJ model: $\mu = J^v_{t+1} = \xi^v_{t+1} = 0$ and $\sqrt{v_t\Delta} = 1$ in the evolution equation.

SP500	Mean	StDev	2.5%	97.5%
$\kappa\theta$	-0.0117	0.0070	-0.0262	0.0014
$1-\kappa$	0.9730	0.0084	0.9551	0.9886
σ^2_v	0.0432	0.0082	0.0302	0.0613
λ	0.0025	0.0017	0.0003	0.0066
μ_y	-2.7254	0.1025	-2.9273	-2.5230
σ^2_y	0.3809	0.2211	0.1445	0.9381

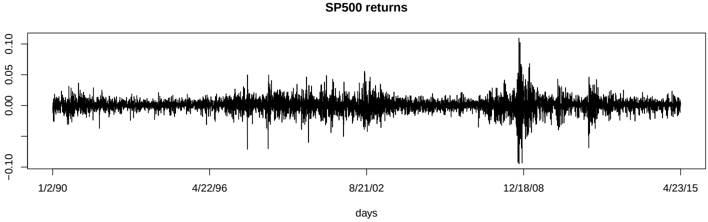
Volatility index - VIX

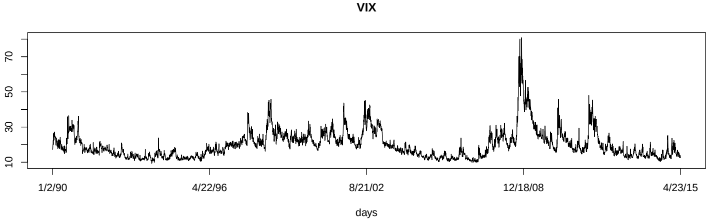
VIX is a trademarked ticker symbol for the Chicago Board Options Exchange (CBOE) Market Volatility Index, a popular measure of the implied volatility of S&P 500 index options.

The VIX is quoted in percentage points and translates, roughly, to the expected movement in the S&P 500 index over the upcoming 30-day period, which is then annualized.

Sources:
`http://en.wikipedia.org/wiki/VIX`
`http://www.cboe.com/micro/vix/vixwhite.pdf`

S&P 500 returns vs VIX





VIX vs model-based volatility