Reescrevendo o ARIMA(1,1,1) no nível

Modelo: \[ \Delta y_t = \alpha + \phi \Delta y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}, \; \epsilon_t \sim (0, \sigma^2) \] ruído branco.

Alternativamente,

\[ y_t - y_{t-1} = \alpha + \phi (y_{t-1} - y_{t-2}) + \epsilon_t + \theta \epsilon_{t-1} \]
\[ = \alpha + \phi y_{t-1} - \phi y_{t-2} + \epsilon_t + \theta \epsilon_{t-1}, \]

ou

\[ y_t = \alpha + (1 + \phi) y_{t-1} - \phi y_{t-2} + \epsilon_t + \theta \epsilon_{t-1}. \]

Portanto, um ARIMA(1,1,1) é um ARMA(2,1)

\[ y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta \epsilon_{t-1}. \]

com a restrição \( \phi_2 = 1 - \phi_1. \)
Previsão no AR(1)

Modelo: \( y_t = \alpha + \phi y_{t-1} + \epsilon_t, \quad \epsilon_t \sim (0, \sigma^2) \) ruído branco.

\[
\begin{align*}
y_{t+1} &= \alpha + \phi y_t + \epsilon_{t+1} \\
y_{t+2} &= \alpha + \phi y_{t+1} + \epsilon_{t+2} \\
&= \alpha + \phi(\alpha + \phi y_t + \epsilon_{t+1}) + \epsilon_{t+2} \\
&= \alpha(1 + \phi) + \phi^2 y_t + \phi \epsilon_{t+1} + \epsilon_{t+2} \\
y_{t+3} &= \alpha + \phi y_{t+2} + \epsilon_{t+3} \\
&= \alpha + \phi[\alpha(1 + \phi) + \phi^2 y_t + \phi \epsilon_{t+1} + \epsilon_{t+2}] + \epsilon_{t+3} \\
&= \alpha(1 + \phi + \phi^2) + \phi^2 y_t + \phi^2 \epsilon_{t+1} + \phi \epsilon_{t+2} + \epsilon_{t+3}
\end{align*}
\]

Portanto,

\[
y_{t+h} = \alpha(1 + \phi + \cdots + \phi^{h-1}) + \phi^{h-1} y_t + \phi^{h-1} \epsilon_{t+1} + \cdots + \phi \epsilon_{t+h-1} + \epsilon_{t+h}
\]

\[
\begin{align*}
E(y_{t+h} | y_t) &= \alpha(1 + \phi + \cdots + \phi^{h-1}) + \phi^{h-1} y_t \\
V(y_{t+h} | y_t) &= \sigma^2(1 + \phi^2 + \phi^4 + \cdots + \phi^{2(h-1)})
\end{align*}
\]
Quando \( \phi \in (-1, 1) \) e \( h \to \infty \),

\[
\phi^h \quad \to \quad 0
\]

\[
(1 + \phi + \phi^2 + \cdots + \phi^{h-1}) \quad \to \quad \frac{1}{1 - \phi}
\]

\[
(1 + \phi^2 + \phi^4 + \cdots + \phi^{2(h-1)}) \quad \to \quad \frac{1}{1 - \phi^2}
\]

e

\[
E(y_{t+h}|y_t) \quad = \quad \frac{\alpha}{1 - \phi}
\]

\[
V(y_{t+h}|y_t) \quad = \quad \frac{\sigma^2}{1 - \phi^2}
\]
Previsão no AR(2)

Modelo: \( y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \quad \epsilon_t \sim (0, \sigma^2). \)

\[
\begin{align*}
y_{t+1} &= \alpha + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1} \\
y_{t+2} &= \alpha + \phi_1 y_{t+1} + \phi_2 y_t + \epsilon_{t+2} \\
&= \alpha + \phi_1 (\alpha + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1}) + \phi_2 y_t + \epsilon_{t+2} \\
&= \alpha (1 + \phi_1) + (\phi_1^2 + \phi_2) y_t + \phi_1 \phi_2 y_{t-1} + \phi_1 \epsilon_{t+1} + \epsilon_{t+2} \\
y_{t+3} &= \alpha (1 + \phi_1 + \phi_1^2 + \phi_2) + (\phi_1^3 + 2\phi_1 \phi_2) y_t + (\phi_1^2 \phi_2 + \phi_2^2) y_{t-1} \\
&+ (\phi_1^2 + \phi_2) \epsilon_{t+1} + \phi_1 \epsilon_{t+2} + \epsilon_{t+3}
\end{align*}
\]

Portanto,

\[
\begin{align*}
V(y_{t+1}|y_{1:t}) &= \sigma^2 \\
V(y_{t+2}|y_{1:t}) &= \sigma^2(1 + \phi_1^2) \\
V(y_{t+3}|y_{1:t}) &= \sigma^2(1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2)
\end{align*}
\]
Previsão 4-passos a frente:

\[
y_{t+4} = \alpha (1 + \phi_1 + \phi_2^2 + \phi_2) \\
+ (\phi_1^3 + 2\phi_1\phi_2)y_{t+1} \\
+ (\phi_1^2\phi_2 + \phi_2^2)y_t \\
+ (\phi_2^2 + \phi_2)\epsilon_{t+2} + \phi_1\epsilon_{t+3} + \epsilon_{t+4}
\]

Como \(y_{t+1} = \alpha + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1}\), segue que

\[
y_{t+3} = \alpha (1 + \phi_1 + \phi_2^2 + \phi_2^3 + \phi_2 + 2\phi_1\phi_2) \\
+ (\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2)y_t \\
+ (\phi_1^2\phi_2 + \phi_2^3)y_{t-1} \\
+ (\phi_1^2\phi_2 + \phi_2^2)\epsilon_{t+1} + (\phi_1^2 + \phi_2)\epsilon_{t+2} + \phi_1\epsilon_{t+3} + \epsilon_{t+4}
\]

e

\[
V(y_{t+4}|y_t) = \sigma^2 \{1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2 + (\phi_1\phi_2 + \phi_2^2)^2\}
\]
Variâncias das previsões

\[
V(y_{t+1}|y_t) = \sigma^2 \\
V(y_{t+2}|y_t) = \sigma^2 \{1 + \phi_1^2\} \\
V(y_{t+3}|y_t) = \sigma^2 \{1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2\} \\
V(y_{t+4}|y_t) = \sigma^2 \{1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2 + (\phi_1 \phi_2 + \phi_2^2)^2\} \\
\vdots \\
V(y_{t+h}|y_t) = \sigma^2 \{1 + \psi_1^2 + \psi_2^2 + \psi_3^2 + \cdots + \psi_{h-1}^2\}
\]