Reescrevendo o ARIMA(1,1,1) no nível

Modelo: $\Delta y_t = \alpha + \phi \Delta y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}$, $\epsilon_t \sim (0, \sigma^2)$ ruído branco.

Alternativamente,

$$\begin{aligned} y_t - y_{t-1} &= \alpha + \phi(y_{t-1} - y_{t-2}) + \epsilon_t + \theta \epsilon_{t-1} \\ &= \alpha + \phi y_{t-1} - \phi y_{t-2} + \epsilon_t + \theta \epsilon_{t-1}, \end{aligned}$$

ou

$$y_t = \alpha + (1+\phi) y_{t-1} - \phi y_{t-2} + \epsilon_t + \theta \epsilon_{t-1}.$$

Portanto, um ARIMA(1,1,1) é um ARMA(2,1)

$$y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta \epsilon_{t-1}.$$

com a restrição $\phi_2 = 1 - \phi_1$.

Previsão no AR(1)

Modelo: $y_t = \alpha + \phi y_{t-1} + \epsilon_t$, $\epsilon_t \sim (0, \sigma^2)$ ruído branco.

$$
\begin{aligned}
y_{t+1} &= \alpha + \phi y_t + \epsilon_{t+1} \\
y_{t+2} &= \alpha + \phi y_{t+1} + \epsilon_{t+2} \\
&= \alpha + \phi(\alpha + \phi y_t + \epsilon_{t+1}) + \epsilon_{t+2} \\
&= \alpha(1+\phi) + \phi^2 y_t + \phi\epsilon_{t+1} + \epsilon_{t+2} \\
y_{t+3} &= \alpha + \phi y_{t+2} + \epsilon_{t+3} \\
&= \alpha + \phi[\alpha(1+\phi) + \phi^2 y_t + \phi\epsilon_{t+1} + \epsilon_{t+2}] + \epsilon_{t+3} \\
&= \alpha(1+\phi+\phi^2) + \phi^2 y_t + \phi^2\epsilon_{t+1} + \phi\epsilon_{t+2} + \epsilon_{t+3}
\end{aligned}
$$

Portanto,

$$y_{t+h} = \alpha(1+\phi+\cdots\phi^{h-1}) + \phi^{h-1} y_t + \phi^{h-1}\epsilon_{t+1} + \cdots + \phi\epsilon_{t+h-1} + \epsilon_{t+h}$$

$$
\begin{aligned}
E(y_{t+h}|y_t) &= \alpha(1+\phi+\cdots\phi^{h-1}) + \phi^{h-1} y_t \\
V(y_{t+h}|y_t) &= \sigma^2(1+\phi^2+\phi^4+\cdots+\phi^{2(h-1)})
\end{aligned}
$$

Quando $\phi \in (-1,1)$ e $h \longrightarrow \infty$,

$$
\begin{aligned}
\phi^h &\longrightarrow 0 \\
(1+\phi+\phi^2+\cdots\phi^{h-1}) &\longrightarrow \frac{1}{1-\phi} \\
(1+\phi^2+\phi^4+\cdots\phi^{2(h-1)}) &\longrightarrow \frac{1}{1-\phi^2}
\end{aligned}
$$

e

$$
\begin{aligned}
E(y_{t+h}|y_t) &= \frac{\alpha}{1-\phi} \\
V(y_{t+h}|y_t) &= \frac{\sigma^2}{1-\phi^2}
\end{aligned}
$$

Previsão no AR(2)

Modelo: $y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \quad \epsilon_t \sim (0, \sigma^2)$.

$$y_{t+1} = \alpha + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1}$$

$$\begin{aligned}
y_{t+2} &= \alpha + \phi_1 y_{t+1} + \phi_2 y_t + \epsilon_{t+2} \\
&= \alpha + \phi_1(\alpha + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1}) + \phi_2 y_t + \epsilon_{t+2} \\
&= \alpha(1 + \phi_1) + (\phi_1^2 + \phi_2) y_t + \phi_1 \phi_2 y_{t-1} + \phi_1 \epsilon_{t+1} + \epsilon_{t+2}
\end{aligned}$$

$$\begin{aligned}
y_{t+3} &= \alpha(1 + \phi_1 + \phi_1^2 + \phi_2) + (\phi_1^3 + 2\phi_1\phi_2) y_t + (\phi_1^2 \phi_2 + \phi_2^2) y_{t-1} \\
&+ (\phi_1^2 + \phi_2)\epsilon_{t+1} + \phi_1 \epsilon_{t+2} + \epsilon_{t+3}
\end{aligned}$$

Portanto,

$$\begin{aligned}
V(y_{t+1}|y_{1:t}) &= \sigma^2 \\
V(y_{t+2}|y_{1:t}) &= \sigma^2(1 + \phi_1^2) \\
V(y_{t+3}|y_{1:t}) &= \sigma^2(1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2)
\end{aligned}$$

Previsão 4-passos a frente:

$$\begin{aligned} y_{t+4} &= \alpha(1+\phi_1+\phi_1^2+\phi_2) \\ &+ (\phi_1^3+2\phi_1\phi_2)y_{t+1} \\ &+ (\phi_1^2\phi_2+\phi_2^2)y_t \\ &+ (\phi^2+\phi_2)\epsilon_{t+2}+\phi_1\epsilon_{t+3}+\epsilon_{t+4} \end{aligned}$$

Como $y_{t+1} = \alpha + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1}$, segue que

$$\begin{aligned} y_{t+3} &= \alpha(1+\phi_1+\phi_1^2+\phi_1^3+\phi_2+2\phi_1\phi_2) \\ &+ (\phi_1^4+3\phi_1^2\phi_2+\phi_2^2)y_t \\ &+ (\phi_1^2\phi_2^2+\phi_2^3)y_{t-1} \\ &+ (\phi_1^2\phi_2+\phi_2^2)\epsilon_{t+1}+(\phi_1^2+\phi_2)\epsilon_{t+2}+\phi_1\epsilon_{t+3}+\epsilon_{t+4} \end{aligned}$$

e

$$V(y_{t+4}|y_t) = \sigma^2\{1+\phi_1^2+(\phi_1^2+\phi_2)^2+(\phi_1^2\phi_2+\phi_2^2)^2\}$$

Variâncias das previsões

$$
\begin{aligned}
V(y_{t+1}|y_t) &= \sigma^2 \\
V(y_{t+2}|y_t) &= \sigma^2\{1+\phi_1^2\} \\
V(y_{t+3}|y_t) &= \sigma^2\{1+\phi_1^2+(\phi_1^2+\phi_2)^2\} \\
V(y_{t+4}|y_t) &= \sigma^2\{1+\phi_1^2+(\phi_1^2+\phi_2)^2+(\phi_1^2\phi_2+\phi_2^2)^2\} \\
&\vdots \\
V(y_{t+h}|y_t) &= \sigma^2\{1+\psi_1^2+\psi_2^2+\psi_3^2+\cdots+\psi_{h-1}^2\}
\end{aligned}
$$