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Question 1. (1.0)

Suppose that $x|\theta \sim N(\theta, 1)$ (for example, x is a measurement of a physical constant θ made with an instrument with variance 1). The prior distribution for θ elicited by the scientist A corresponds to a $N(5, 1)$ distribution and the scientist B elicits a $N(15, 1)$ distribution. The value $x = 6$ was observed. What prior fits the data better?

Question 2. (1.0)

Consider the following longitudinal data model:

$$\begin{aligned} y_{ti}|x_{ti}, \alpha_i, \sigma^2 &\sim N(\alpha_i x_{ti}, \sigma^2) \\ \alpha_i|\alpha, \tau^2 &\sim N(\alpha, \tau^2), \end{aligned}$$

where y_{it} refers to the outcomes for individual (or more generally, group) i at time i and α_i is a person-specific random (slope) effect. We assume $i = 1, 2, \ldots, N$ and $t = 1, 2, \ldots, T$ (i.e., a balanced panel). Derive the conditional posterior distribution $p(\alpha_i|\alpha, \sigma^2, \tau^2, y, x)$.

Question 3. (1.0)

Let $x|\theta, \mu \sim N(\theta, \sigma^2)$, σ^2 known, $\theta|\mu \sim N(\mu, \tau^2)$, τ^2 known and $\mu \sim N(0, 1)$. Obtain the following distributions:

(a) $(\theta|x, \mu)$

(b) $(\mu|x)$

(c) $(\theta|x)$

Question 4. (1.0)

Count data is a typical kind of discrete data, where the observations y_i are equal to zero or a positive integer; that is, $y_i \in \{0, 1, \dots, \}$. Such data are often modeled by the Poisson distribution,

$$p(y_i|\theta_i) = \frac{e^{-\theta_i}\theta_i^{y_i}}{y_i!} \qquad \text{where } \theta_i = \exp\{-x_i'\beta\}.$$

Find an expression for the posterior distribution, $p(\beta|y)$, on the assumption that $p(\beta) \equiv N(\beta_0, B_0)$, and discuss possible ways to simulate from this distribution.

Question 5. (2.0)

The random variable x has double exponential distribution with parameters μ and σ, denoted by $DE(\mu, \sigma)$, if its density is

$$f(x|\mu,\sigma) = \frac{1}{2\sigma}\exp\left\{-\frac{|x-\mu|}{\sigma}\right\}, \quad \text{for } x \in \mathbb{R}\,.$$

Show that if $x|y \sim N(0, y)$ and $y \sim Exp(1/2)$ then $x \sim DE(0, 1)$.

Question 6. (2.0)

Suppose that the density for a time series y_t, $t = 1, 2, \dots, T$, conditioned on its lags, the model parameters, and other covariates, can be expressed as

$$y_t|\beta_1, \beta_2, \sigma_1^2, \sigma_2^2, \tau, x_t \sim \begin{cases} N(\beta_1 x_t, \sigma_1^2) & t \le \tau \\ N(\beta_2 x_t, \sigma_2^2) & t > \tau \end{cases}$$

In this model, τ is a changepoint: For periods until τ, ie. $t \le \tau$, one regression is assumed to generate y, and following τ, ie. $t > \tau$, a new regression is assumed to generate y. Suppose you employ priors of the form

$$\begin{array}{rcll} \beta_1 & \sim & N(b_{01}, B_{01}) & \quad \beta_2 \sim N(b_{02}, B_{02}) \\ \sigma_1^2 & \sim & IG(c_{01}, d_{01}) & \quad \sigma_2^2 \sim IG(c_{02}, d_{02}) \\ \tau & \sim & U\{1, 2, \dots, T-1\}. & \end{array}$$

Note that τ is treated as a parameter of the model, and by placing a uniform prior over the elements $1, 2, \dots, T-1$ a changepoint is assumed to exist. Describe how the Gibbs sampler can be employed to estimate the parameters of this model.

Question 7. (2.0)

Consider the simple regression model

$$y_i^*|\beta, \sigma^2 \sim N(\beta x_i, \sigma^2).$$

Assume that x_i is observed for all $i = 1, 2, \ldots, n$, and that y_i^* is observed (and equal to y_i) for $i = 1, 2, \ldots, m$, where $m < n$. For $i = m+1, m+2, \ldots, n$, the dependent variable is missing, and is presumed to be generated by the above model. The missing data are missing completely at random. Let $N(b_0, B_0)$ and $IG(c, d)$ be the prior distributions of β and σ^2, respectively. Describe a data augmentation approach for *filling in* the missing data and fitting the regression model.

Useful results

- $x \sim N(\mu, \sigma^2)$ when

$$p(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad \text{for } x, \mu \in \mathbb{R}, \sigma^2 > 0.$$

- $x \sim IG(a, b)$ when

$$p(x|a, b) = \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{-b/x}, \qquad \text{for } x, a, b > 0.$$

- Andrews and Mallows (1974)[1] show that

$$\int_0^\infty \exp\{-0.5(a^2u^2 + b^2u^{-2})\}du = \left(\frac{\pi}{2a^2}\right)^{1/2} \exp\{-|ab|\}.$$

[1] Andrews and Mallows (1974) Scale mixtures of normal distributions. *JRSS-B*, **36**, 99-102. See also, West (1987) On scale mixtures of normal distributions. *Biometrika*, **74**, 646-648.