Bayesian Econometrics | **Summer Quarter 2013**
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Suppose we are using an Exponential(λ) distribution[1] to model the lifetimes of n items, ie. the lifetimes $t_1, \ldots, t_n$ are conditionally independent and identically distributed Exponential(λ). In addition, suppose $\gamma \sim Gamma(a, b)$ summarizes the prior[2] information regarding λ. It is also relatively straightforward to see that $(s_n|\lambda) \sim Gamma(n, \lambda)$, where $s_n = \sum_{i=1}^n t_i$. Derive

a) The maximum likelihood estimator of λ, ie. $\hat{\lambda}$.

b) The standard error of $\hat{\lambda}$ (assuming n is large).

c) The 90% confidence interval (L, U) for λ.

d) The prior predictive[3] for s_n, $p(s_n)$.

e) The posterior distribution for λ, ie. $p(\lambda|s_n)$.

f) The posterior $p(t_{n+1}|s_n)$.

Suppose that we observed $n = 50$ items and that $s_{50} = 25$. Also, let the prior hyperparameters of the prior be $a = 1$ and $b = 2$.

g) Compare the MLE summaries from a) and b) to posterior summaries $E(\lambda|s_{50} = 25)$ and $\sqrt{V(\lambda|s_{50} = 25)}$, respectively.

h) Plot the posterior $p(\lambda|s_{50} = 25)$ and posterior predictive $p(t_{51}|s_{50} = 25)$.

f) What is the posterior probability that λ falls in the 90% confidence interval found in (c)? Or, $Pr(L \leq \lambda \leq U|s_{50} = 25)$?

[1] We say that the random variable T has an exponential distribution (or is an exponential random variable), and write $T \sim Exponential(\lambda)$ if the probability density function (pdf) for T is $p(t|\lambda) = \lambda \exp\{-\lambda t\}$ for $t > 0$ and $\lambda > 0$. $E(t|\lambda) = 1/\lambda$ and $V(t|\lambda) = 1/\lambda^2$.

[2] We say that $\lambda \sim Gamma(a, b)$ if its pdf is $p(\lambda|a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp\{-b\lambda\}$. $E(\lambda|a, b) = a/b$ and $V(\lambda|a, b) = a/b^2$. In fact, $Gamma(1, b) \equiv Exp(b)$.

[3] Also known as *predictive*, *normalizing constant*, or *marginal likelihood*.