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Lecture 2: Bayesian Computation

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MC in the 40s and 50s

Stan Ulam soon realized that computers could be used in this fashion to answer questions of neutron diffusion and mathematical physics;

He contacted John Von Neumann and they developed many Monte Carlo algorithms (importance sampling, rejection sampling, etc);

In the 1940s Nick Metropolis and Klari Von Neumann designed new controls for the state-of-the-art computer (ENIAC);

Metropolis and Ulam (1949) The Monte Carlo method. *Journal of the American Statistical Association*.
Metropolis *et al.* (1953) Equations of state calculations by fast computing machines. *Journal of Chemical Physics*.

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Monte Carlo methods

We introduce several Monte Carlo (MC) methods for integrating and/or sampling from nontrivial densities.

- MC integration
  - Simple MC integration
  - MC integration via importance sampling (IS)
- MC sampling
  - Rejection method
  - Sampling importance resampling (SIR)
- Iterative MC sampling
  - Metropolis-Hastings algorithms
  - Simulated annealing
  - Gibbs sampler

Based on the book by Gamerman and Lopes (1996).

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A few references

- MC integration (Geweke, 1989)
- Rejection methods (Gilks and Wild, 1992)
- SIR (Smith and Gelfand, 1992)
- Metropolis-Hastings algorithm (Hastings, 1970)
- Simulated annealing (Metropolis *et al.*, 1953)
- Gibbs sampler (Gelfand and Smith, 1990)

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Two main tasks

1. Compute high dimensional integrals:

$$E_\pi[h(\theta)] = \int h(\theta)\pi(\theta)d\theta$$

2. Obtain

$$a \ sample \ \{\theta_1, \ldots, \theta_n\} \ from \ \pi(\theta)$$

when only

$$a \ sample \ \{\tilde{\theta}_1, \ldots, \tilde{\theta}_m\} \ from \ q(\theta)$$

is available.

$q(\theta)$ is known as the *proposal/auxiliary* density.

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Bayes via MC

MC methods appear frequently, but not exclusively, in modern Bayesian statistics.

Posterior and predictive densities are hard to sample from:

$$\text{Posterior} \quad : \quad \pi(\theta) = \frac{f(x|\theta)p(\theta)}{f(x)}$$

$$\text{Predictive} \quad : \quad f(x) = \int f(x|\theta)p(\theta)d\theta$$

Other important integrals and/or functionals of the posterior and predictive densities are:

- Posterior modes: $\max_\theta \pi(\theta)$;
- Posterior moments: $E_\pi[g(\theta)]$;
- Density estimation: $\hat{\pi}(g(\theta))$;
- Bayes factors: $f(x|M_0)/f(x|M_1)$;
- Decision: $\max_d \int U(d,\theta)\pi(\theta)d\theta$.

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MC integration

The objective here is to compute moments

$$E_\pi[h(\theta)] = \int h(\theta)\pi(\theta)d\theta$$

If $\theta_1, \ldots, \theta_n$ is a random sample from $\pi(\cdot)$ then

$$\bar{h}_{mc} = \frac{1}{n}\sum_{i=1}^{n} h(\theta_i) \rightarrow E_\pi[h(\theta)] \qquad \text{as } n \rightarrow \infty.$$

If, additionally, $E_\pi[h^2(\theta)] < \infty$, then

$$V_\pi[\bar{h}_{mc}] = \frac{1}{n}\int \{h(\theta) - E_\pi[h(\theta)]\}^2\pi(\theta)d\theta$$

and

$$v_{mc} = \frac{1}{n^2}\sum_{i=1}^{n}(h(\theta_i) - \bar{h}_{mc})^2 \rightarrow V_\pi[\bar{h}_{mc}] \qquad \text{as } n \rightarrow \infty.$$

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Example i. MC integration

The objective here is to compute[1]

$$p = \int_0^1 [cos(50\theta) + sin(20\theta)]^2 d\theta$$

by noticing that the above integral can be rewritten as

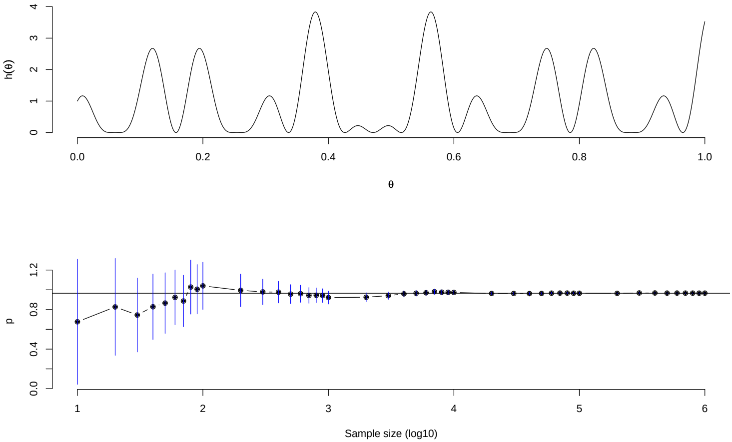
$$E_\pi[h(\theta)] = \int h(\theta)\pi(\theta)d\theta$$

where $h(\theta) = [cos(50\theta) + sin(20\theta)]^2$ and $\pi(\theta) = 1$ is the density of a $U(0,1)$. Therefore

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} h(\theta_i)$$

where $\theta_1, \ldots, \theta_n$ are i.i.d. from $U(0,1)$.

[1] True value is 0.965.


h(θ)
0
1
2
3
4
0.0
0.2
0.4
0.6
0.8
1.0
θ
p
0.0
0.4
0.8
1.2
1
2
3
4
5
6
Sample size (log10)

MC via IS

The objective is still the same, ie to compute

$$E_\pi[h(\theta)] = \int h(\theta)\pi(\theta)d\theta$$

by noticing that

$$E_\pi[h(\theta)] = \int \frac{h(\theta)\pi(\theta)}{q(\theta)} q(\theta)d\theta$$

where $q(\cdot)$ is an *importance function*.

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If $\theta_1, \ldots, \theta_n$ is a random sample from $q(\cdot)$ then

$$\Rightarrow \bar{h}_{is} = \frac{1}{n} \sum_{i=1}^{n} \frac{h(\theta_i)\pi(\theta_i)}{q(\theta_i)} \rightarrow E_\pi[h(\theta)]$$

as $n \longrightarrow \infty$.

Ideally, $q(\cdot)$ should be

- As *close* as possible to $h(\cdot)\pi(\cdot)$, and
- Easy to sample from.

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Example ii. Cauchy tail

The objective here is to estimate

$$p = Pr(\theta > 2) = \int_2^\infty \frac{1}{\pi(1+\theta^2)} d\theta = 0.1475836$$

where θ is a standard Cauchy random variable.

A natural MC estimator of p is

$$\hat{p}_1 = \frac{1}{n} \sum_{i=1}^{n} I\{\theta_i \in (2, \infty)\}$$

where $\theta_1, \ldots, \theta_n \sim$ Cauchy(0,1).

A more elaborated estimator based on a change of variables from θ to $u = 1/\theta$ is

$$\hat{p}_2 = \frac{1}{n}\sum_{i=1}^{n} \frac{u_i^{-2}}{2\pi[1 + u_i^{-2}]}$$

where $u_1, \ldots, u_n \sim U(0, 1/2)$.

The true value is $p = 0.147584$.

n	$\hat{p}_1$	$\hat{p}_2$	$v_1^{1/2}$	$v_2^{1/2}$
100	0.100000	0.1467304	0.030000	0.001004
1000	0.137000	0.1475540	0.010873	0.000305
10000	0.148500	0.1477151	0.003556	0.000098
100000	0.149100	0.1475591	0.001126	0.000031
1000000	0.147711	0.1475870	0.000355	0.000010

With only $n = 1000$ draws, $\hat{p}_2$ has roughly the same precision that $\hat{p}_1$, which is based on $1000n$ draws, ie. three orders of magnitude.

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Rejection method

The objective is to draw from a target density

$$\pi(\theta) = c_\pi \tilde{\pi}(\theta)$$

when only draws from an auxiliary density

$$q(\theta) = c_q \tilde{q}(\theta)$$

is available, for normalizing constants c_π and c_q.

If there exist a constant $A < \infty$ such that

$$0 \leq \frac{\tilde{\pi}(\theta)}{A\tilde{q}(\theta)} \leq 1 \quad \text{for all } \theta$$

then $q(\theta)$ becomes a *blanketing density* or an *envelope* and A the *envelope constant*.

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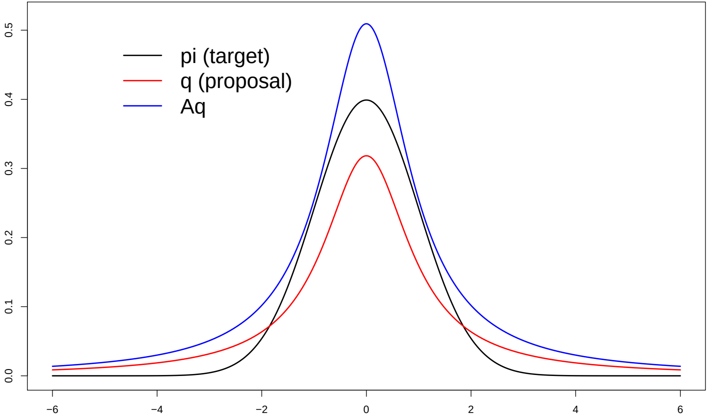
Blanket distribution



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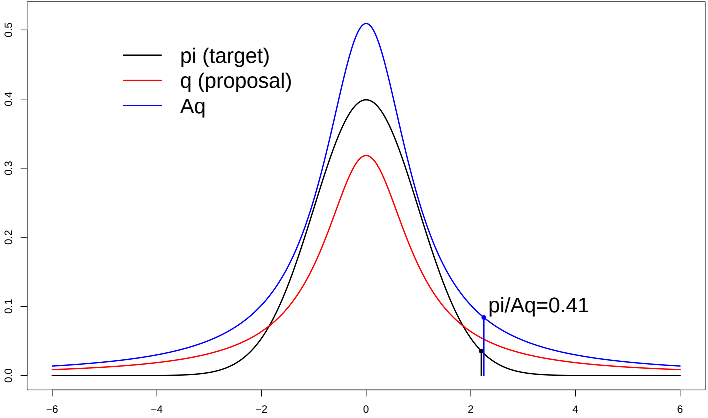
Bad draw



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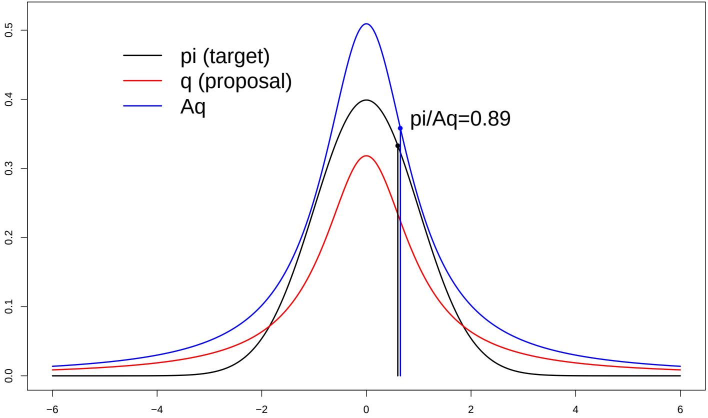
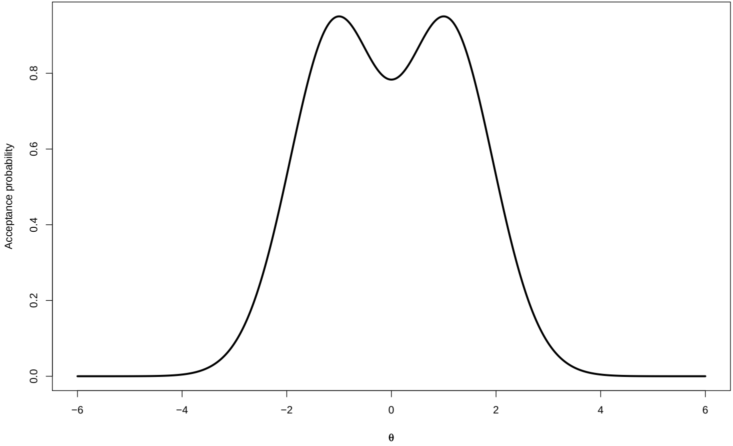
Good draw



Acceptance probability



Algorithm

Drawing from $\pi(\theta)$.

1. Draw θ^* from $q(\cdot)$;
2. Draw u from $U(0,1)$;
3. Accept θ^* if $u \leq \frac{\tilde{\pi}(\theta^*)}{A\tilde{q}(\theta^*)}$;
4. Repeat 1, 2 and 3 until n draws are accepted.

Normalizing constants c_π and c_q are not needed.

The theoretical acceptance rate is $\frac{c_q}{Ac_\pi}$.

The smaller the A, the larger the acceptance rate.

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Example iii. Sampling $N(0,1)$

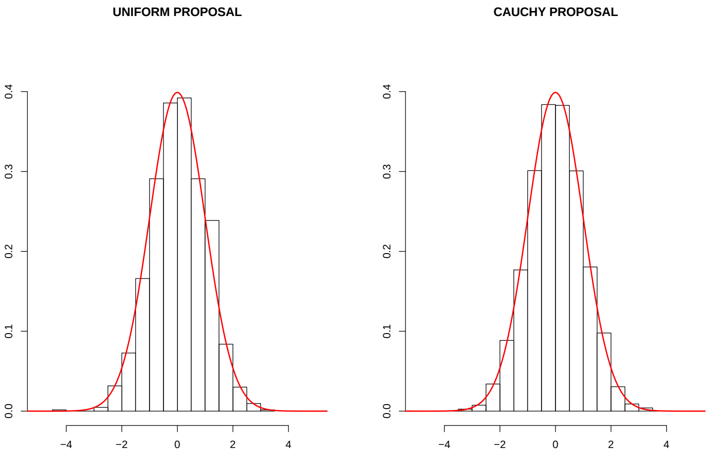
Enveloping the standard normal density

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}} \exp\{-0.5\theta^2\}$$

by a Cauchy density $q_C(\theta) = 1/(\pi(1+\theta^2))$, or a uniform density $q_U(\theta) = 0.05$ for $\theta \in (-10, 10)$.

Bad proposal:The maximum of $\pi(\theta)/q_U(\theta)$ is roughly $A_U = 7.98$ for $\theta \in (-10, 10)$. The theoretical acceptance rate is 12.53%.

Good proposal: The max of $\pi(\theta)/q_C(\theta)$ is equal to $A_C = \sqrt{2\pi/e} \approx 1.53$. The theoretical acceptance rate is 65.35%.



Empirical rates: 0.1265 (Uniform) and 0.6483 (Cauchy)
Theoretical rates: 0.1253 (Uniform) and 0.6535 (Cauchy)

SIR method

No need to rely on the existance of A!

Algorithm

1. Draw $\theta_1^*, \ldots, \theta_n^*$ from $q(\cdot)$
2. Compute (unnormalized) weights

$$\omega_i = \pi(\theta_i^*)/q(\theta_i^*) \qquad i = 1, \ldots, n$$

3. Sample θ from $\{\theta_1^*, \ldots, \theta_n^*\}$ such that

$$Pr(\theta = \theta_i^*) \propto \omega_i \qquad i = 1, \ldots, n.$$

4. Repeat m times step 3.

Rule of thumb: $n/m = 20$.
Ideally, $\omega_i = 1/n$ and $Var(\omega) = 0$.

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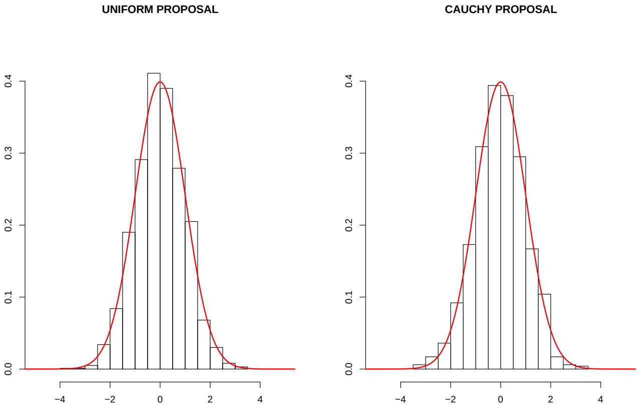
Example iii. revisited



Fraction of redraws: 0.391 (Uniform) and 0.1335 (Cauchy)
Variance of weights: 4.675 (Uniform) and 0.332 (Cauchy)

Example iv. 3-component mixture

Assume that we are interested in sampling from

$$\pi(\theta) = \alpha_1 p_N(\theta; \mu_1, \Sigma_1) + \alpha_2 p_N(\theta; \mu_2, \Sigma_2) + \alpha_3 p_N(\theta; \mu_3, \Sigma_3)$$

where $p_N(\cdot; \mu, \Sigma)$ is the density of a bivariate normal distribution with mean vector μ and covariance matrix Σ. The mean vectors are

$$\mu_1 = (1,4)' \quad \mu_2 = (4,2)' \quad \mu_3 = (6.5,2),$$

the covariance matrices are

$$\Sigma_1 = \begin{pmatrix} 1.0 & -0.9 \\ -0.9 & 1.0 \end{pmatrix} \text{ and } \Sigma_2 = \Sigma_3 = \begin{pmatrix} 1.0 & -0.5 \\ -0.5 & 1.0 \end{pmatrix},$$

and weights $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$.

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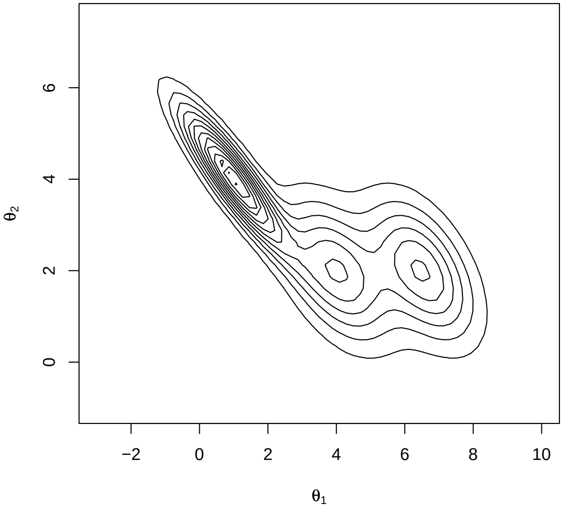
Target $\pi(\theta)$



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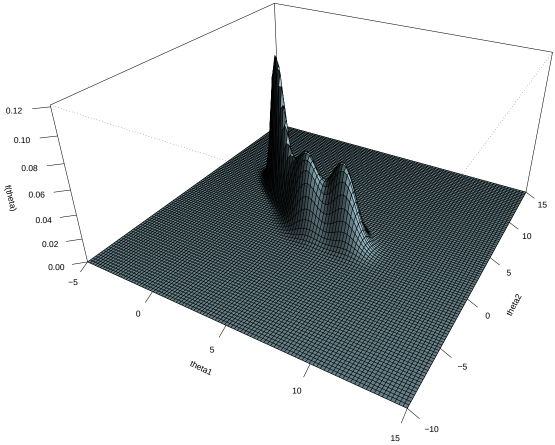
Target $\pi(\theta)$



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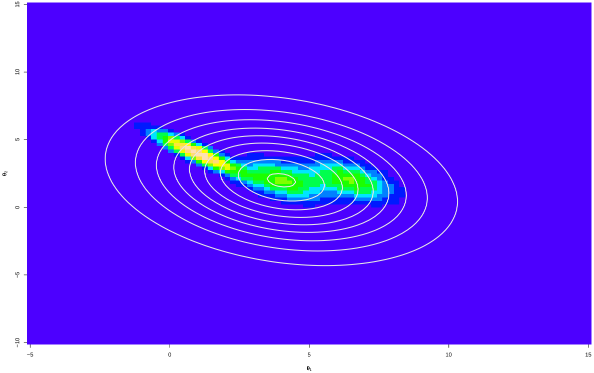
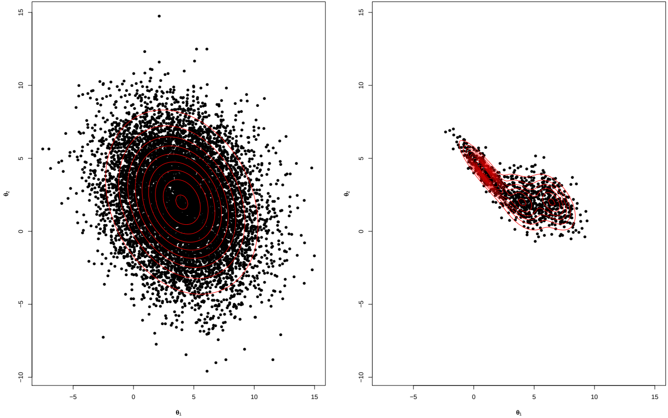
Proposal $q(\theta)$



$q(\theta) \sim N(\mu, \Sigma)$ where

$$\mu_2 = (4,2)' \quad \text{and} \quad \Sigma = 9\begin{pmatrix} 1.0 & -0.25 \\ -0.25 & 1.0 \end{pmatrix}$$

Rejection method



Acceptance rate: 9.91% of $n = 10,000$ draws.

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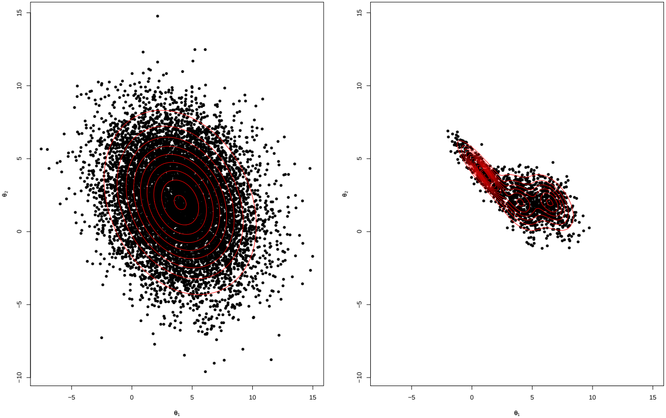
SIR method



Fraction of redraws: 29.45% of ($n = 10,000$, $m = 2,000$).

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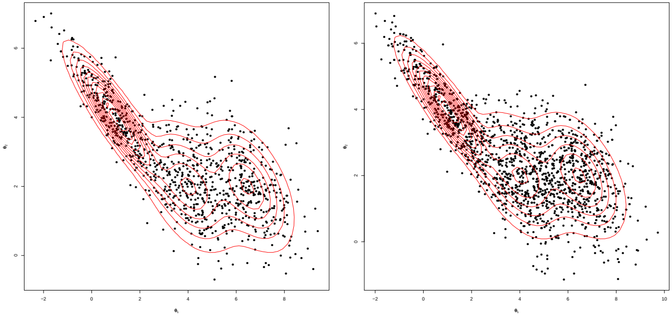
Rejection & SIR



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Example v. 2-component mixture

Let us now assume that

$$\pi(\theta) = \alpha_1 p_N(\theta; \mu_1, \Sigma_1) + \alpha_3 p_N(\theta; \mu_3, \Sigma_3)$$

where mean vectors are

$$\mu_1 = (1, 4)' \quad \mu_3 = (6.5, 2),$$

the covariance matrices are

$$\Sigma_1 = \begin{pmatrix} 1.0 & -0.9 \\ -0.9 & 1.0 \end{pmatrix} \text{ and } \Sigma_3 = \begin{pmatrix} 1.0 & -0.5 \\ -0.5 & 1.0 \end{pmatrix},$$

and weights $\alpha_1 = 1/3$ and $\alpha_3 = 2/3$.

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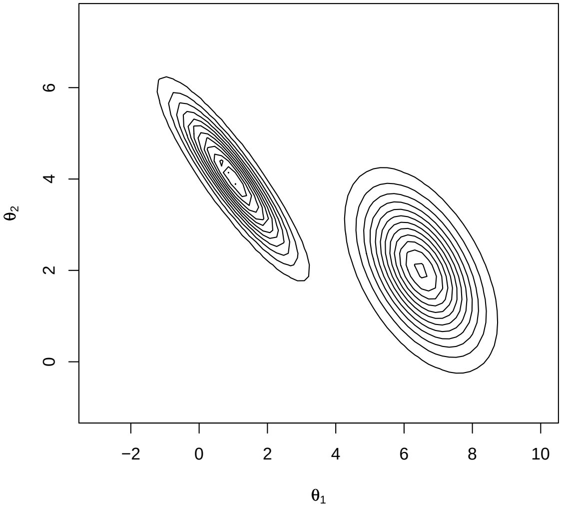
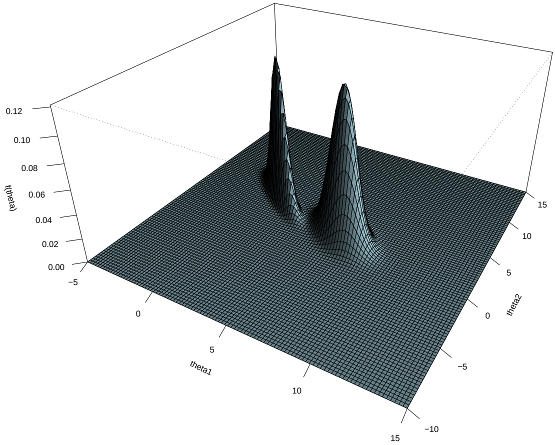
Target $\pi(\theta)$



Target $\pi(\theta)$



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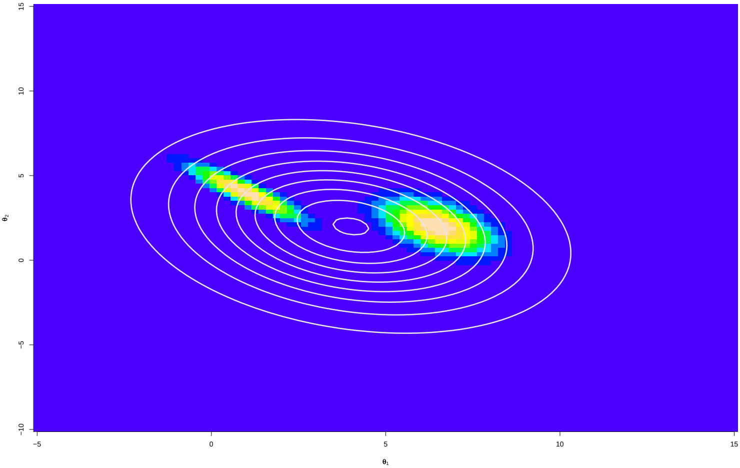
Proposal $q(\theta)$



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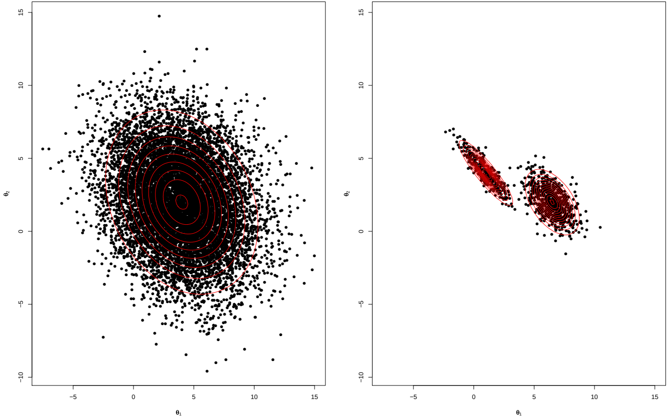
Rejection method



Acceptance rate: 10.1% of $n = 10,000$ draws.

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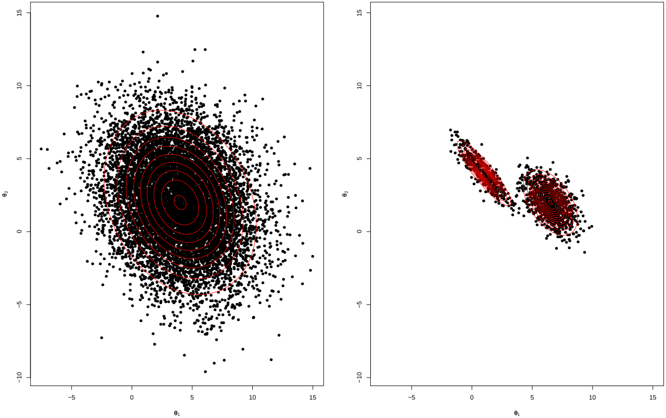
SIR method



Fraction of redraws: 37.15% of $(n = 10,000, m = 2,000)$.

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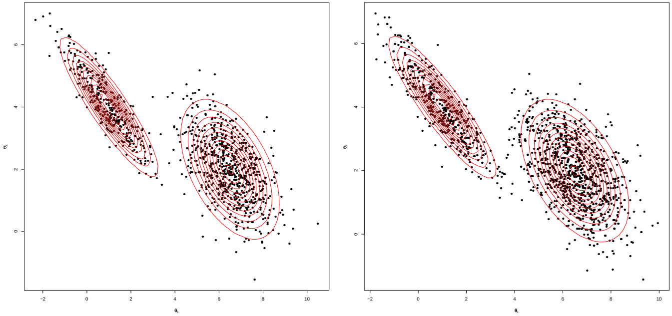
Rejection & SIR



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MCMC history

Dongarra and Sullivan (2000) list the top algorithms with the greatest influence on the development and practice of science and engineering in the 20th century (in chronological order):

- Metropolis Algorithm for Monte Carlo
- Simplex Method for Linear Programming
- Krylov Subspace Iteration Methods
- The Decompositional Approach to Matrix Computations
- The Fortran Optimizing Compiler
- QR Algorithm for Computing Eigenvalues
- Quicksort Algorithm for Sorting
- Fast Fourier Transform

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70s and 80s

Metropolis-Hastings:
Hastings (1970) and his student Peskun (1973) showed that Metropolis and the more general Metropolis-Hastings algorithm are particular instances of a larger family of algorithms.

Gibbs sampler:

Besag (1974) Spatial Interaction and the Statistical Analysis of Lattice Systems.
Geman and Geman (1984) Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images.
Pearl (1987) Evidential reasoning using stochastic simulation.
Tanner and Wong (1987). The calculation of posterior distributions by data augmentation.
Gelfand and Smith (1990) Sampling-based approaches to calculating marginal densities.

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MH algorithms

A sequence $\{\theta^{(0)}, \theta^{(1)}, \theta^{(2)}, \ldots\}$ is drawn from a Markov chain whose *limiting equilibrium distribution* is the posterior distribution, $\pi(\theta)$.

Algorithm

1. Initial value: $\theta^{(0)}$
2. Proposed move: $\theta^* \sim q(\theta^*|\theta^{(i-1)})$
3. Acceptance scheme:

$$\theta^{(i)} = \begin{cases} \theta^* & \text{com prob.} \quad \alpha \\ \theta^{(i-1)} & \text{com prob.} \quad 1-\alpha \end{cases}$$

where

$$\alpha = \min\left\{1, \frac{\pi(\theta^*)}{\pi(\theta^{(i-1)})} \frac{q(\theta^{(i-1)}|\theta^*)}{q(\theta^*|\theta^{(i-1)})}\right\}$$

Special cases

1. Symmetric chains: $q(\theta|\theta^*) = q(\theta^*|\theta)$

$$\alpha = \min\left\{1, \frac{\pi(\theta^*)}{\pi(\theta)}\right\}$$

2. Independence chains: $q(\theta|\theta^*) = q(\theta)$

$$\alpha = \min\left\{1, \frac{\omega(\theta^*)}{\omega(\theta)}\right\}$$

where $\omega(\theta^*) = \pi(\theta^*)/q(\theta^*)$.

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Random walk Metropolis

The most famous symmetric chain is the random walk Metropolis:

$$q(\theta|\theta^*) = q(|\theta - \theta^*|)$$

Hill climbing: when

$$\alpha = \min\left\{1, \frac{\pi(\theta^*)}{\pi(\theta)}\right\}$$

a value θ^* with higher density $\pi(\theta^*)$ greater than $\pi(\theta)$ is automatically accepted.

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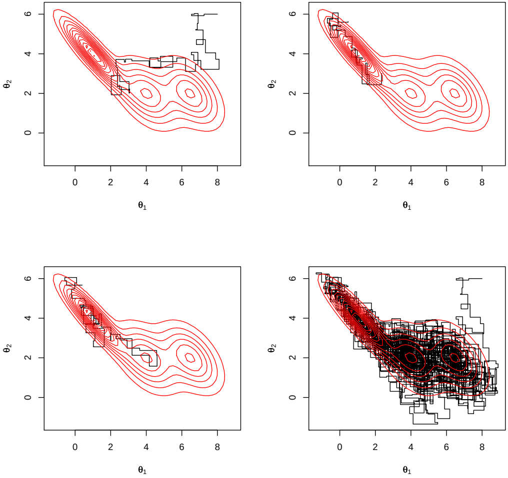
Example iv. RW Metropolis



$q(\theta|\theta_i) \sim N(\theta_i, 0.25\Sigma_2)$.

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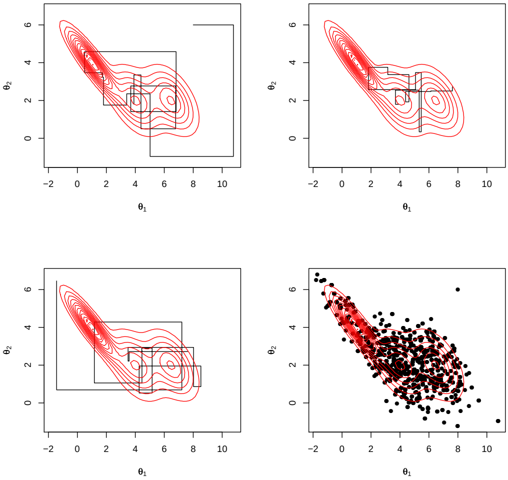
Example iv. Ind. Metropolis



$$q(\theta) \equiv q_{SIR}(\theta) \sim N(\mu, \Sigma).$$

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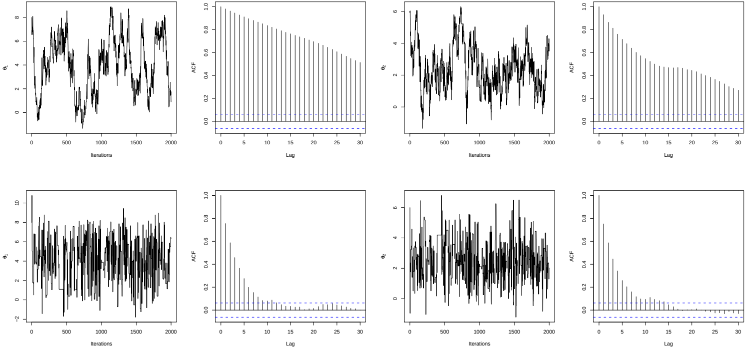
Example iv. Autocorrelations



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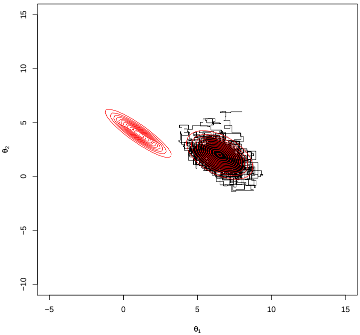
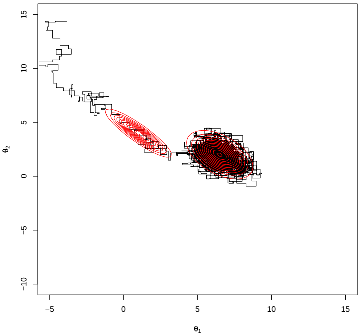
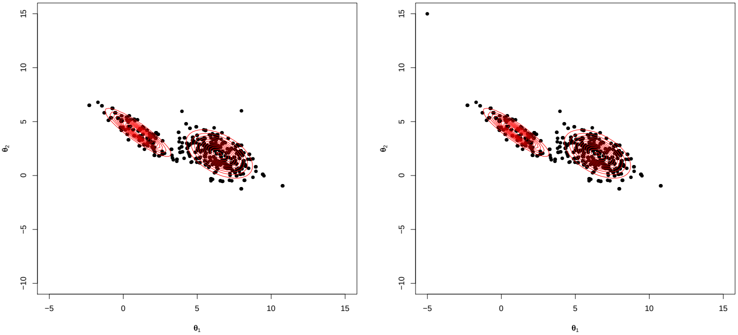
Example v. RW Metropolis

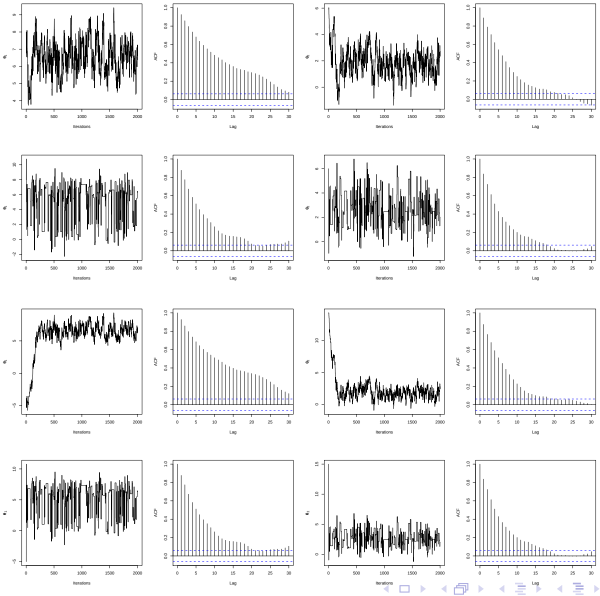




Example v. Ind. Metropolis



Example v. Autocorrelations



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Example vi. tuning selection

Tthe target distribution is a two-component mixture of bivariate normal densities, ie:

$$\pi(\theta) = 0.7 f_N(\theta; \mu_1, \Sigma_1) + 0.3 f_N(\theta; \mu_2, \Sigma_2).$$

where

$$\begin{aligned}
\mu_1' &= (4.0, 5.0) \\
\mu_2' &= (0.7, 3.5) \\
\Sigma_1 &= \begin{pmatrix} 1.0 & 0.7 \\ 0.7 & 1.0 \end{pmatrix} \\
\Sigma_2 &= \begin{pmatrix} 1.0 & -0.7 \\ -0.7 & 1.0 \end{pmatrix}.
\end{aligned}$$

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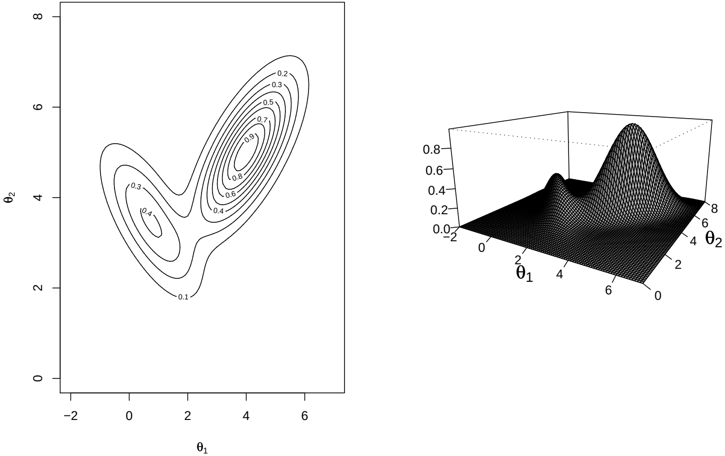
Target distribution



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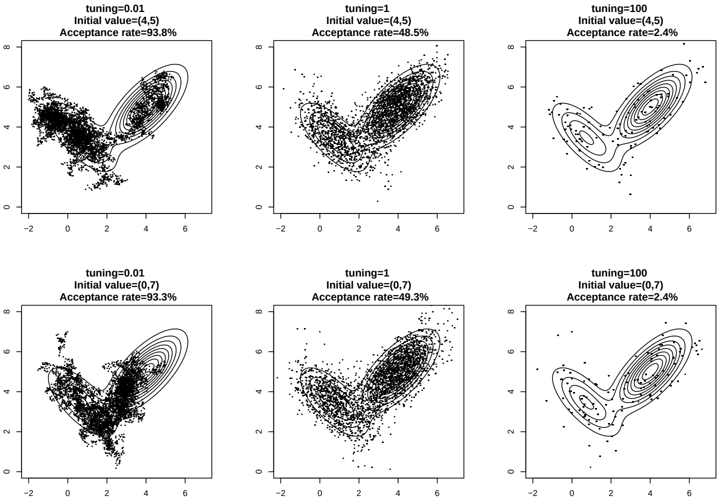
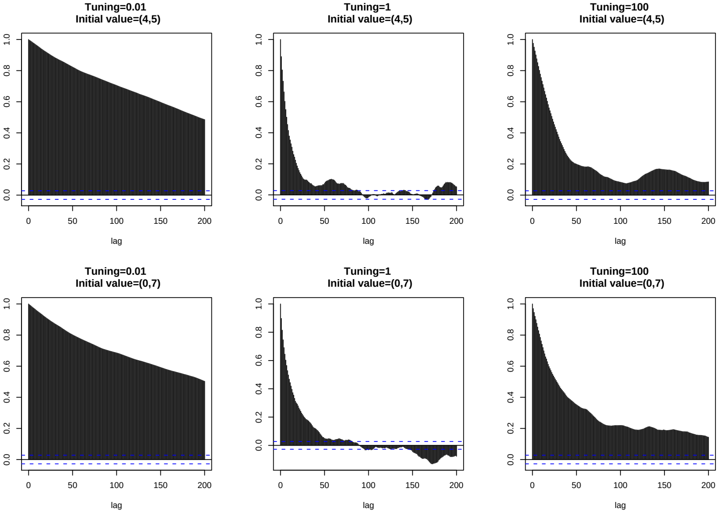
RW Metropolis

$q(\theta,\phi) = f_N(\phi;\theta,\nu I_2)$ and ν =tuning.

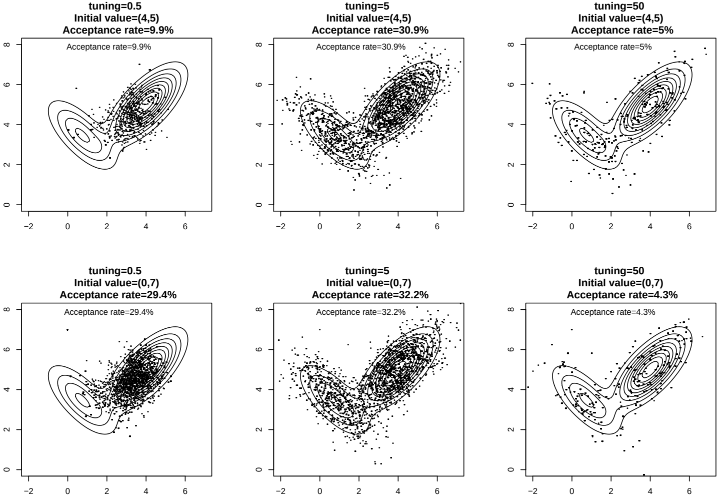


Autocorrelations



Independent Metropolis

$q(\theta, \phi) = f_N(\phi; \mu_3, \nu I_2)$ and $\mu_3 = (3.01, 4.55)'$.



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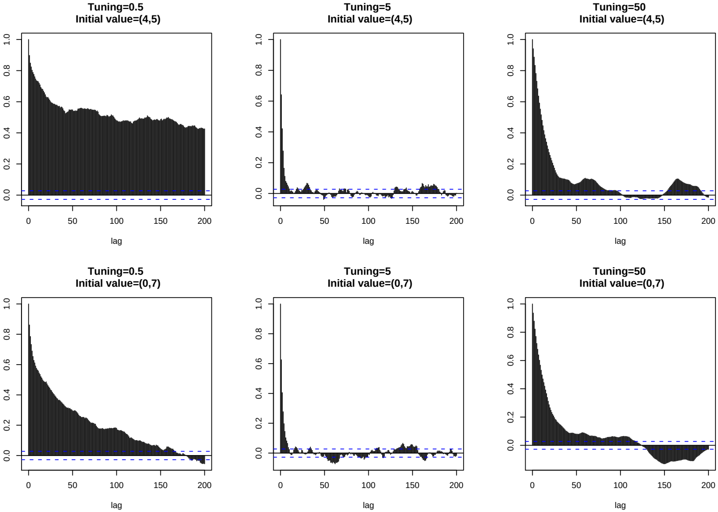

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Simulated annealing

Simulated annealing[2] is an optimization technique designed to find maxima of functions.

It can be seen as a M-H algorithm that *tempers* with the target distribution:

$$q(\theta) \propto \pi(\theta)^{1/T}$$

where the constant $T > 1$ receives the physical interpretation of system temperature, hence the nomenclature used (Jennison, 1993).

The *heated* distribution q is flattened with respect to π and its density gets closer to the uniform distribution, which is particularly relevant for the case of a distribution with distant modes.

By flattening the modes, the moves required to cover adequately the parameter space become more likely.

[2] Kirkpatrick, Gelatt and Vecchi (1983)

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Example vii: Nonlinear surface

Assume that the goal is to find the mode/maximum of

$$\pi(\beta_1, \beta_2) \propto \prod_{i=1}^{4} \frac{e^{(\beta_1+\beta_2 x_i) y_i}}{(1 + e^{\beta_1+\beta_2 x_i})^5},$$

with $x = (-0.863, -0.296, -0.053, 0.727)$ and $y = (0, 1, 3, 5)$.

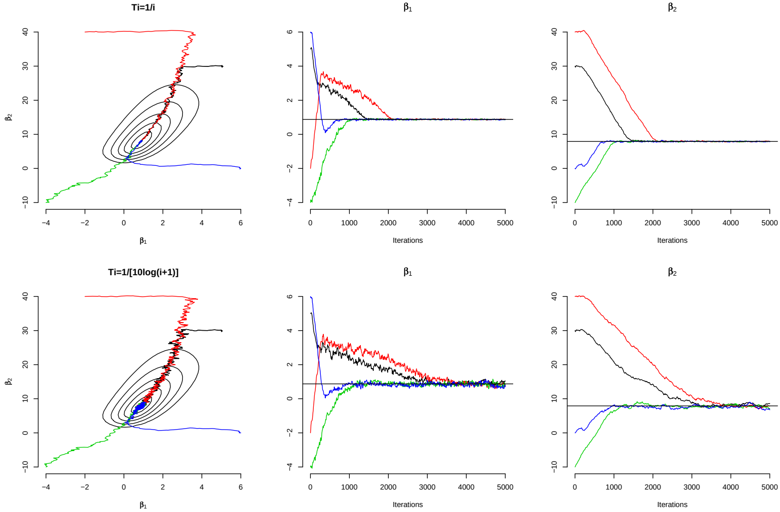
The simulated annealing algorithm is implemented for four initial values:

$$(5, 30) \quad (-2, 40) \quad (-4, -10) \quad (6, 0)$$

and two cooling schedules:

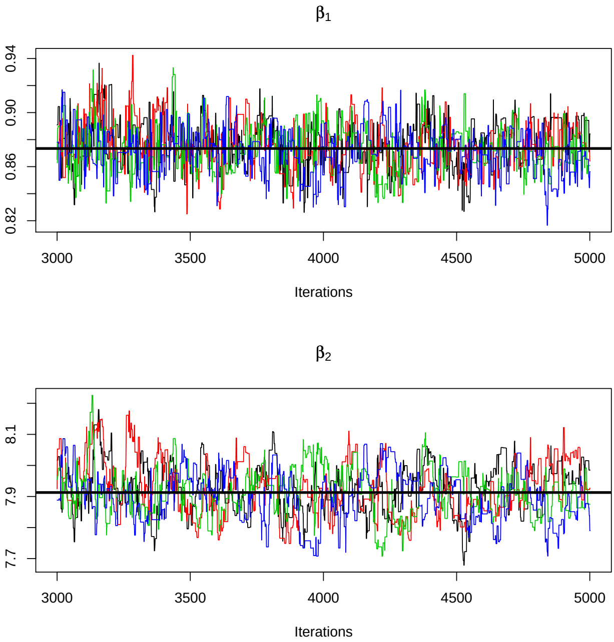
$$T_i = 1/i \quad \text{and} \quad T_i = 1/[10 \log(1 + i)].$$

The proposal distribution is $q(\beta|\beta^{(i)}) = f_N(\beta; \beta^{(i)}, 0.05^2 I_2)$.



Newton-Raphson mode: $(0.87, 7.91)$.

$T_i = 1/i$: mode is $(0.88, 7.99)$ when $(\beta_1^{(0)}, \beta_2^{(0)}) = (5, 30)$.


β1
0.94
0.90
0.86
0.82
3000
3500
4000
4500
5000
Iterations
β2
8.1
7.9
7.7
3000
3500
4000
4500
5000
Iterations

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Gibbs sampler

Technically, the Gibbs sampler is an MCMC scheme whose transition kernel is the product of the full conditional distributions.

Algorithm

1. Start at $\theta^{(0)} = (\theta_1^{(0)}, \theta_2^{(0)}, \ldots)$
2. Sample the components of $\theta^{(j)}$ iteratively:

$$\begin{array}{rcl}
\theta_1^{(j)} & \sim & \pi(\theta_1 | \theta_2^{(j-1)}, \theta_3^{(j-1)}, \ldots) \\
\theta_2^{(j)} & \sim & \pi(\theta_2 | \theta_1^{(j)}, \theta_3^{(j-1)}, \ldots) \\
\theta_3^{(j)} & \sim & \pi(\theta_3 | \theta_1^{(j)}, \theta_2^{(j)}, \ldots) \\
 & \vdots &
\end{array}$$

The Gibbs sampler opened up a new way of approaching statistical modeling by combining simpler structures (the full conditional models) to address the more general structure (the full model).

Example viii: Bivariate normal

Assume that the target distribution is the bivariate normal with mean vector and covariance matrix given by

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix},$$

respectively.

In this case, the two full conditionals are given by

$$\theta_1|\theta_2 \sim N\left(\mu_1 + \frac{\sigma_{12}}{\sigma_2^2}(\theta_2 - \mu_2), \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right)$$

and

$$\theta_2|\theta_1 \sim N\left(\mu_2 + \frac{\sigma_{12}}{\sigma_1^2}(\theta_1 - \mu_1), \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}\right)$$

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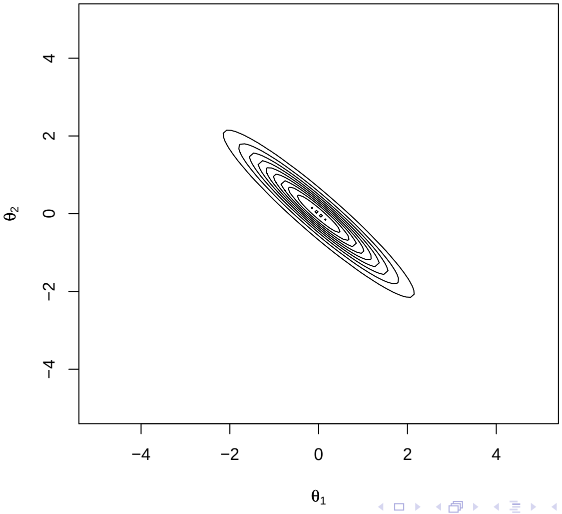
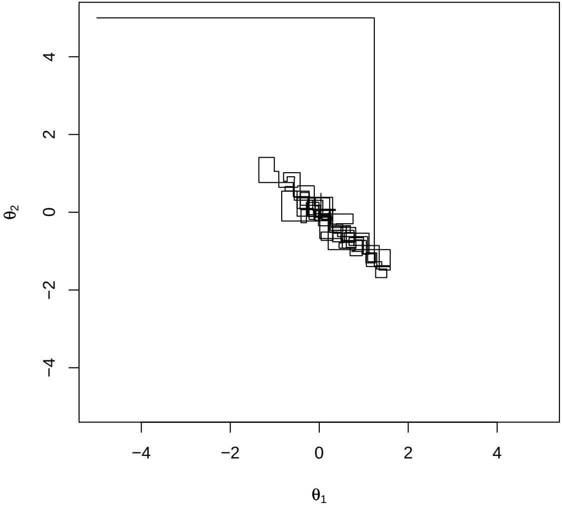
$$\mu_1 = \mu_2 = 0$$
$$\sigma_1^2 = \sigma_2^2 = 1$$
$$\sigma_{12} = -0.95$$




4
2
0
−2
−4
θ_2
−4
−2
0
2
4
θ_1

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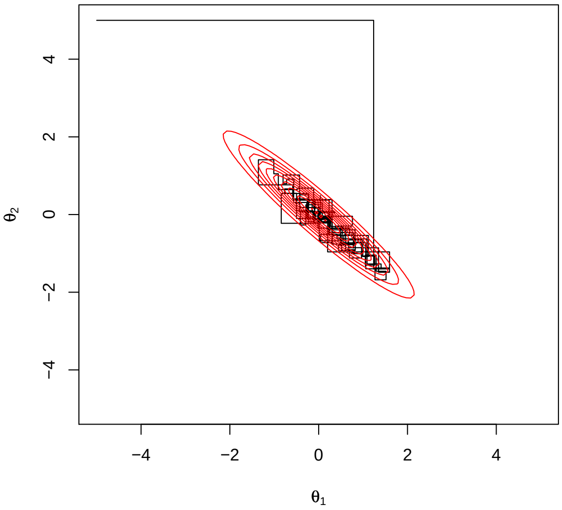
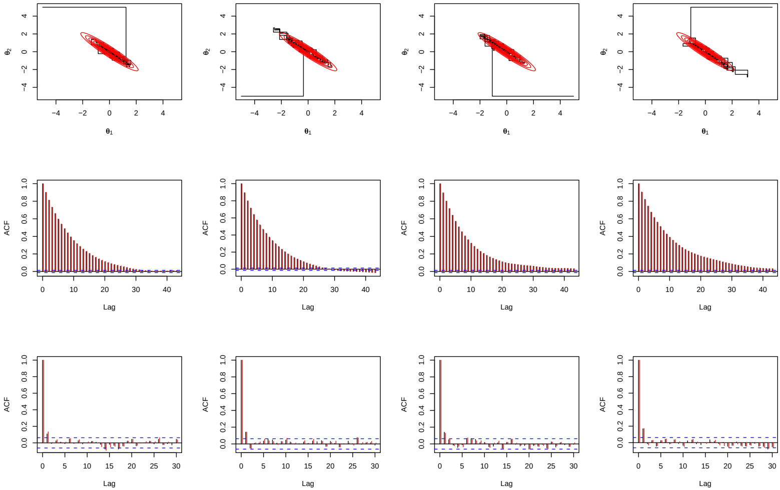
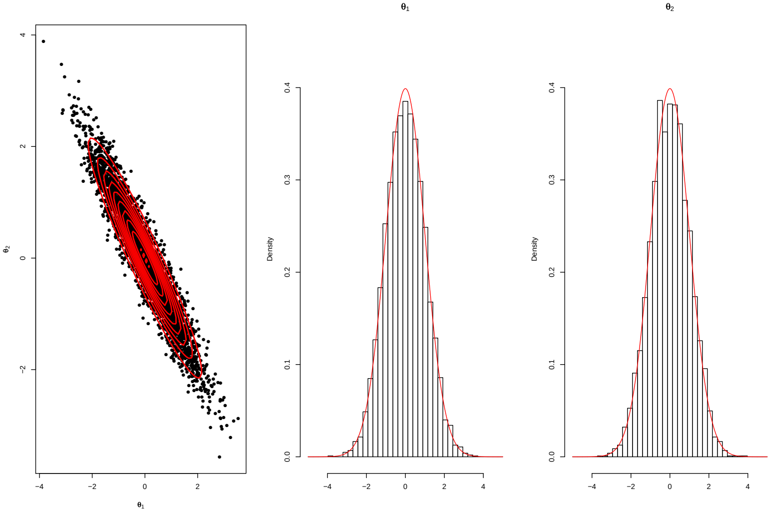




Middle frame: Based on $M = 21,000$ consecutive draws.
Bottom frame: Based on $M = 1000$ draws, after initial $M_0 = 1000$ draws and saving every 20th draws.


θ₁
θ₂
θ₂
θ₁
Density
Density
-4
-2
0
2
4
0.0
0.1
0.2
0.3
0.4

Example: Seemingly Unrelated Regressions

Investments

Grunfeld (1958), Boot and White (1960) and Zellner (1962,1963) study

$$I_{mt} = \beta_{m1} + \beta_{m2} F_{mt} + \beta_{m3} C_{mt} + \varepsilon_{mt}$$

- Should the parameters be the same across firms?
- Do the ε_{mt} share unobserved common factors?
- Staking the observation for firm m:

$$y_m = X_m \beta_m + \varepsilon_m$$

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Capital asset pricing

For a given security, the CAPM specifies that

$$r_{mt} - r_{ft} = \alpha_m + \beta_m(r_{mt} - r_{ft}) + \varepsilon_{mt}$$

for return on security m, r_{mt}, return on a risk-free security, r_{ft}, and market return, r_{mt}.

- Are the disturbances correlated across securities?
- Are the α_ms and/or β_ms related in any way?
- Staking the observation for security m:

$$y_m = X_m\beta_m + \varepsilon_m$$

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Gross State Product

Greene (2008) examines (his examples 9.9 and 9.12) Munnell's (1990) model for output by the 48 continental US states:

$$\begin{aligned} \log GSP_{mt} &= \beta_{m1} + \beta_{m2} \log pcap_{mt} + \beta_{m3} \log hwy_{mt} \\ &+ \beta_{m4} \log water_{mt} + \beta_{m5} \log util_{mt} \\ &+ \beta_{m6} \log emp_{mt} + \beta_{m7} unemp_{mt} + \varepsilon_{mt} \end{aligned}$$

- Should the coefficient vector be the same across states?
- Should the disturbances correlated across states?
- Should the disturbances correlated across time?
- Staking the observation for state m:

$$y_m = X_m \beta_m + \varepsilon_m$$

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For $m = 1, \ldots, M$ and $t = 1, \ldots, T$

$$y_{mt} = x'_{mt}\beta_m + \epsilon_{mt},$$

with x_{mt} a k_m-dimensional vector of regressors.

Let us stack all equations:

$$\begin{array}{rcll} y_t & = & (y_{1t}, \ldots, y_{Mt})' & (M \times 1) \\ \varepsilon_t & = & (\varepsilon_{1t}, \ldots, \varepsilon_{Mt})' & (M \times 1) \\ \beta & = & (\beta'_1, \ldots, \beta'_M)' & (k \times 1) \\ X_t & = & \text{diag}(x'_{1t}, \ldots, x'_{Mt}) & (M \times k) \end{array}$$

where $k = \sum_{m=1}^{M} k_m$. Therefore,

$$y_t = X_t\beta + \varepsilon_t$$

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We can now stack all observations $t = 1, \ldots, T$ together:

$$\begin{aligned} y &= (y_1, \ldots, y_T)' \\ \varepsilon &= (\varepsilon_t, \ldots, \varepsilon_T)' \\ X &= (X_1', \ldots, X_T')', \end{aligned}$$

such that

$$y = X\beta + \varepsilon.$$

NLRM: ε_{mt} are i.i.d. $N(0, \sigma^2)$ for all m and t.

SUR: ε_t are i.i.d. $N(0, \Sigma)$ for all t.

This leads to $\varepsilon \sim N(0, \Omega)$, where

$$\Omega = \text{diag}(\Sigma, \ldots, \Sigma) = I_T \otimes \Sigma$$

is an $MT \times MT$ block-diagonal covariance matrix.

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Prior distribution

Conditionally conjugate prior for β and $\Phi = \Sigma^{-1}$:

$$p(\beta, \Phi) = p(\beta)p(\Phi),$$

where

$$\beta \sim N(\beta_0, V_0)$$

and

$$\Phi \sim Wishart(\nu_0, \Phi_0).$$

See Dreze and Richard (1983) and Richard and Steel (1988) for further discussion regarding alternative prior specifications.

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Full conditionals

The full conditional distributions are

$$\begin{aligned} \beta|y, X, \Sigma &\sim N(\beta_1, V_1) \\ \Phi|y, X, \beta &\sim Wishart(\nu_1, \Phi_1) \end{aligned}$$

where $\nu_1 = \nu_0 + T$,

$$\begin{aligned} V_1^{-1} &= V_0^{-1} + \sum_{t=1}^{T} X_t' \Phi X_t \\ V_1^{-1}\beta_1 &= V_0^{-1}\beta_0 + \sum_{t=1}^{n} X_t' \Phi y_t \\ \Phi_1^{-1} &= \Phi_0^{-1} + \sum_{t=1}^{T} (y_t - X_t\beta)(y_t - X_t\beta)' \end{aligned}$$

Grunfeld's (1958) data

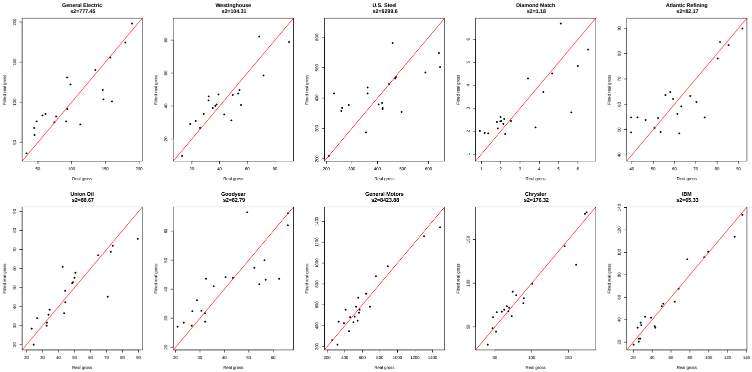
M=10 U.S. firms over T=20 years, 1935-1954.[3]

Variables:
FN = Firm Number; YR = Year;
I = Annual real gross investment;
F = Real value of the firm (shares outstanding); and
C = Real value of the capital stock.

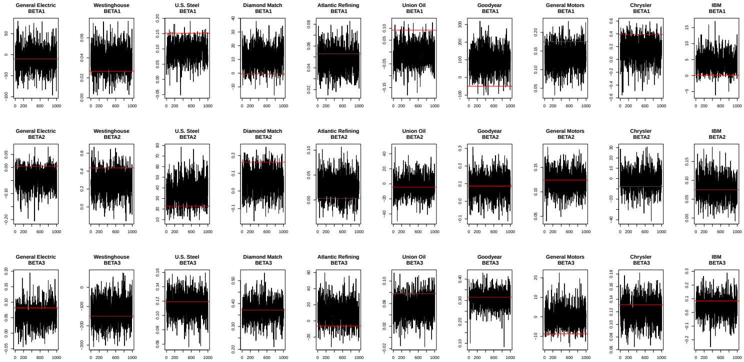
Firms:
General Electric, Westinghouse,
U.S. Steel, Diamond Match,
Atlantic Refining, Union Oil,
Goodyear, General Motors,
Chrysler and IBM

[3] Zellner (1971), pages 240-246.

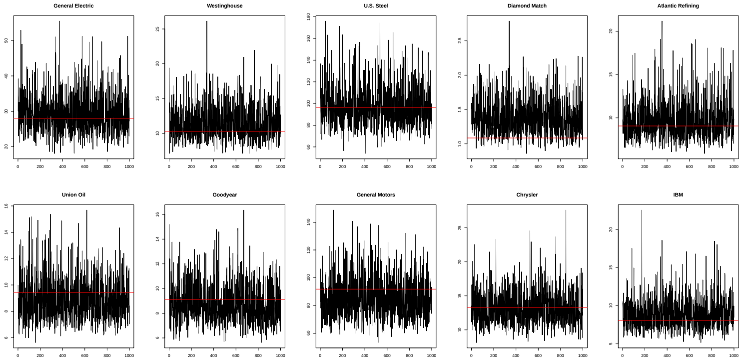
Individual regressions



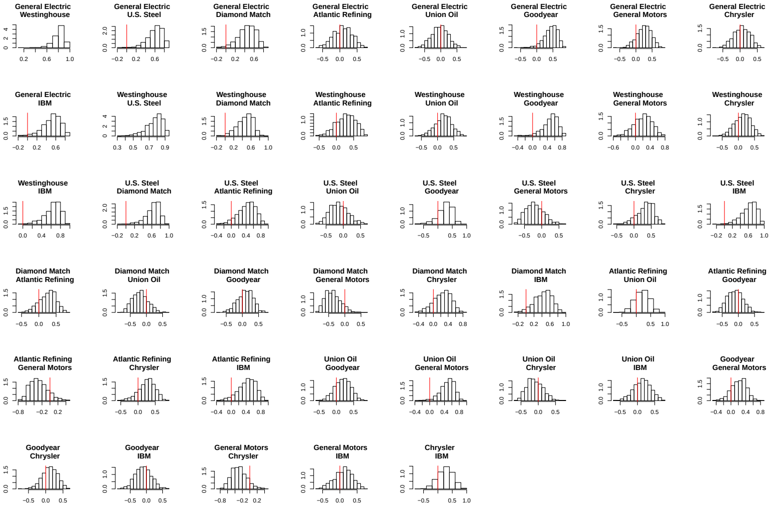
Regression coefficients



Standard deviations



Correlations



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Boot and De Wit (1960) Investment Demand: An Empirical Contribution to the Aggregation. Problem, *International Economic Review*, 1, 3-30.

Zellner (1971) *An Introduction to Bayesian Inference and Econometrics*, Wiley.

Koop (2003) *Bayesian Econometrics*, Wiley.

Kleiber and Zeileis (2010) The Grunfeld Data at 50, *German Economic Review*, 11(4), 404-417.

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Example: Bivariate BVAR(1)

Let $y_t = (y_{t1}, y_{t2})'$ contain 2 time series observed at time t.

The (basic) VAR(1) can be written as

$$(y_t | y_{t-1}, B, \Sigma) \sim N(By_{t-1}, \Sigma)$$

where

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

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VAR(1) as a SUR model

The above VAR(1) model can be rewritten as a SUR model as

$$(y_t|z_t, \beta, \Sigma) \sim N(z_t\beta, \Sigma)$$

where

$$z_t = \begin{pmatrix} y_{t-1,1} & y_{t-1,2} & 0 & 0 \\ 0 & 0 & y_{t-1,1} & y_{t-1,2} \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{21} \\ \beta_{22} \end{pmatrix}.$$

Therefore,

$$(y|\beta, \Sigma) \sim N(Z\beta, \Sigma)$$

where $y = (y_1', \ldots, y_T')'$ and $Z = (z_1', \ldots, z_T')'$.

Prior of (β, Σ)

We assume that β and Σ are independent *a priori*.

Prior of β:

$$\beta \sim N(b_0, B_0).$$

Prior of Σ:

$$\Sigma \sim IW(v_0, V_0).$$

This conditionally conjugate prior DOES NOT lead to closed form posterior inference, but the implementation of the Gibbs sampler is straightforward.

Full conditional of β

It is easy to see that

$$\begin{aligned} p(\beta|\Sigma, y) &\propto \exp\left\{-0.5\left[\beta' B_0^{-1}\beta - 2\beta' B_0^{-1}\beta_0\right]\right\} \\ &\times \exp\left\{-0.5\left[\beta' Z'\Sigma^{-1} Z\beta - 2\beta' Z'\Sigma^{-1} y\right]\right\}. \end{aligned}$$

Therefore,

$$\beta|\Sigma, \gamma, y \sim N(\beta_1, V_1)$$

where

$$\beta_1 = B_1(B_0^{-1}\beta_0 + Z'\Sigma^{-1}y) \quad \text{and} \quad B_1^{-1} = B_0^{-1} + Z'\Sigma^{-1}Z.$$

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Full conditional of Σ

It is easy to see that

$$\begin{aligned} p(\Sigma|\beta, y) &\propto |\Sigma|^{\frac{q+v_0+1}{2}} \exp\left\{-0.5\text{tr}(\Sigma^{-1}V_0)\right\} \\ &\times |\Sigma|^{\frac{q+T+1}{2}} \exp\left\{-0.5\sum_{t=1}^{T}(y_t - z_t\beta)'\Sigma^{-1}(y_t - z_t\beta)\right\}. \end{aligned}$$

Therefore,

$$\Sigma|\beta, y \sim IW(v_1, V_1)$$

where $v_1 = v_0 + T$ and

$$S = V_0 + \sum_{t=1}^{T}(y_t - z_t\beta)(y_t - z_t\beta)'.$$

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Simulated data

We simulated $n = 360$ observations (30 years of monthly data) from the above bivariate VAR(1) with

$$B = \begin{pmatrix} 0.85 & 0.10 \\ 0.00 & 0.95 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} 1.0 & 0.2 \\ 0.2 & 1.0 \end{pmatrix}$$

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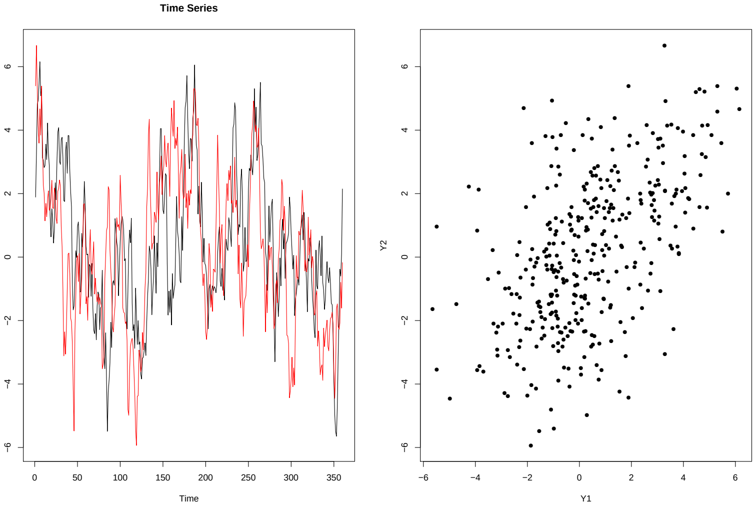


Posterior inference

The prior hyperparameters are

$$b_0 = 0_4 \quad \text{and} \quad B_0 = 1000 I_4$$

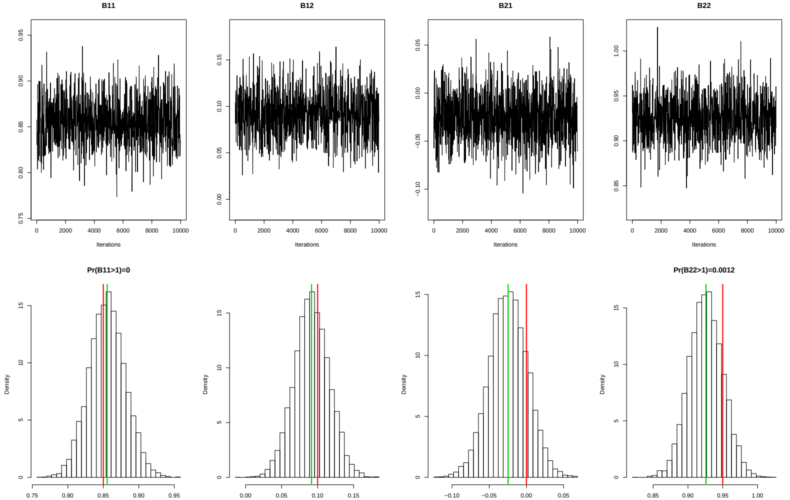
and

$$v_0 = 5 \quad \text{and} \quad V_0 = 0.001$$

We started the Gibbs sampler with $B^{(0)} = B$ (true value).

We runt he Gibbs sampler for $M = 10,000$ iterations.

$p(B|\text{data})$



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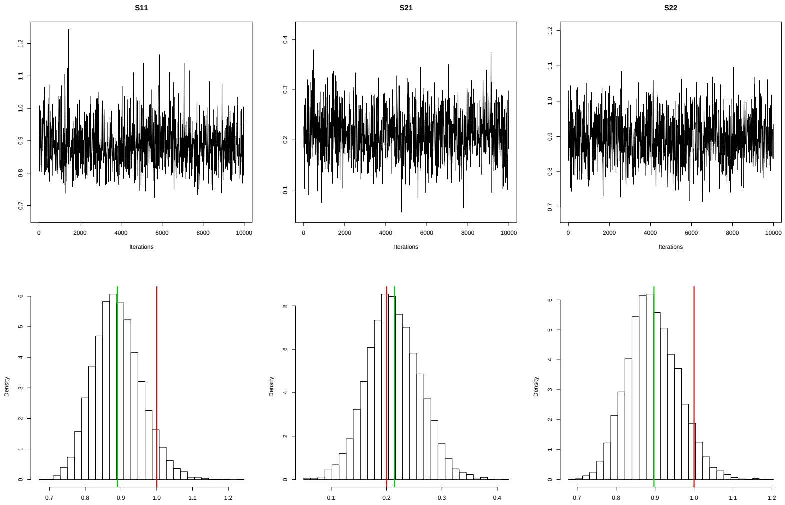
$p(\Sigma|\text{data})$



A bit of history
MC integration
MC via IS
Rejection method
SIR method
MH algorithms
Simulated annealing
Gibbs sampler
SUR model
BVAR
References

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