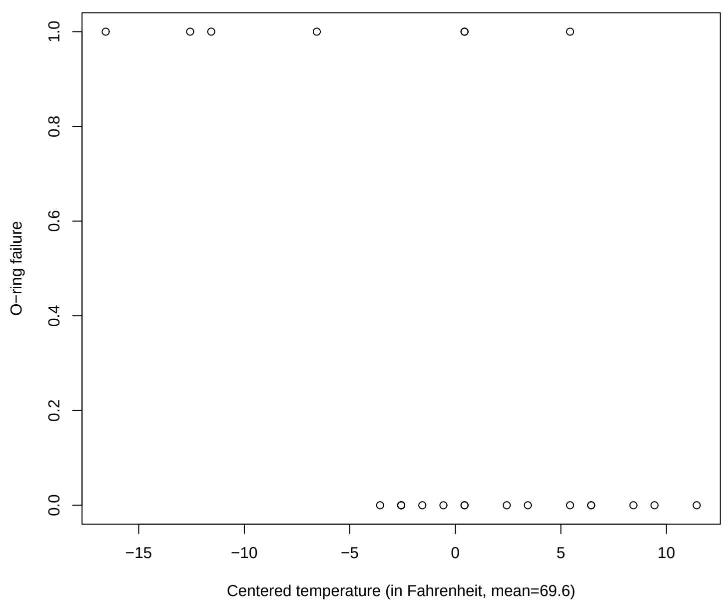
Example: O-ring failures by temperature

3 link functions: logit,probit, log-log

- Christensen (1997) and Congdon (2001) analyze 23 binary observations of O-ring failures y_i (1=failure) in relation to temperature t_i (Fahrenheit).
- $y = (1,1,1,1,0,0,0,0,0,0,0,0,1,1,0,0,0,1,0,0,0,0,0)$
- $t = (53,57,58,63,66,67,67,67,68,69,70,70,70,$ $70,72,73,75,75,76,76,78,79,81)$



What is $Pr(\tilde{y} = 1|\tilde{x})$, for $\tilde{x} = 31, 33, \ldots, 51$?

- Bernoulli model:

$$y_i|\theta_i \sim Bern(\theta_i)$$

for $i = 1, \ldots, n = 23$.

- Link function

  – Logit link (M_1):

$$\log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

  – Probit link (M_2):

$$\Phi(\theta_i) = \alpha + \beta x_i$$

  – Complementary log-log link (M_3):

$$\log(-\log(1-\theta_i)) = \alpha + \beta x_i$$

- $\alpha = -1.26$, $x_i = t_i - \bar{t}$ and $\bar{t} = 69.6$.

- Kernel of the posterior of β

$$
\begin{aligned}
p(\beta|y, M_j) &\propto p(\beta|M_j) l_j(\theta(\beta); y) \\
&\propto e^{-\frac{(\beta-\beta_0)^2}{2V_\beta}} \prod_{i=1}^{n} \theta_i^{y_i}(1-\theta_i)^{1-y_i}
\end{aligned}
$$

  where θ_is are deterministic functions of β and j indexes the corresponding link function, for $j = 1, 2, 3$.

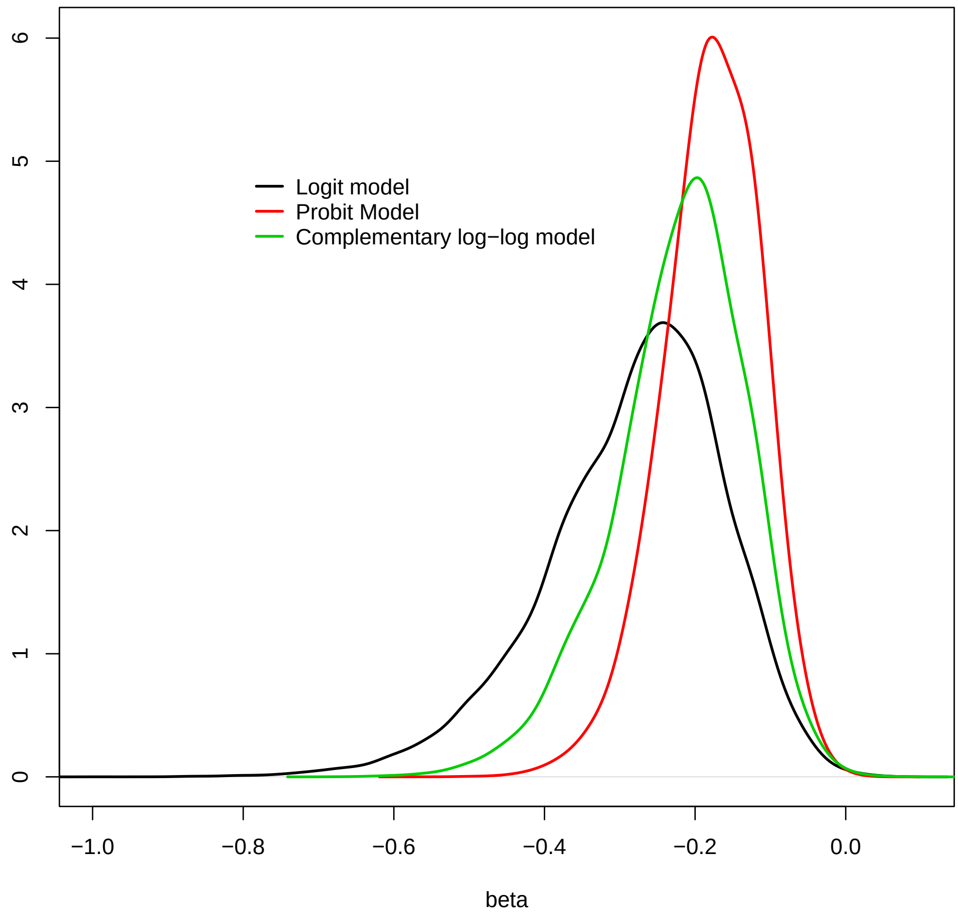
- If $\beta_1, \ldots, \beta_{50000}$ is a sample from $p(\beta)$, then

$$
\begin{aligned}
\widehat{p}(y|M_1) &= 0.000001298981 \\
\widehat{p}(y|M_2) &= 0.000000120205 \\
\widehat{p}(y|M_3) &= 0.000001469835
\end{aligned}
$$

  are MC estimates of $p(y|M_j)$ for $j = 1, 2, 3$.

- The data supports more (higher $p(y|M)$) the logit (M_1) and complementary log-log (M_3) links.

Posterior distribution of β under the three link functions



Bayesian model averaging

$$
\begin{aligned}
Pr(\tilde{y}=1|\tilde{x}) &= \pi_1 \int_{-\infty}^{\infty} Pr(\tilde{y}=1|\tilde{x},\beta,M_1)p(\beta|y,M_1)d\beta \\
&+ \pi_2 \int_{-\infty}^{\infty} Pr(\tilde{y}=1|\tilde{x},\beta,M_2)p(\beta|y,M_2)d\beta \\
&+ \pi_3 \int_{-\infty}^{\infty} Pr(\tilde{y}=1|\tilde{x},\beta,M_3)p(\beta|y,M_3)d\beta
\end{aligned}
$$

where $\pi_j = Pr(M_j|y)$, for $j = 1, 2, 3$, are posterior model probabilities and can be approximated by

$$\widehat{\pi}_j = \frac{\widehat{p}(y|M_j)Pr(M_j)}{\sum_{l=1}^{3} \widehat{p}(y|M_l)Pr(M_l)}$$

For simplicity, the prior model probabilities, $Pr(M_j)$, are set to $1/3$ for $j = 1, 2, 3$. Therefore,

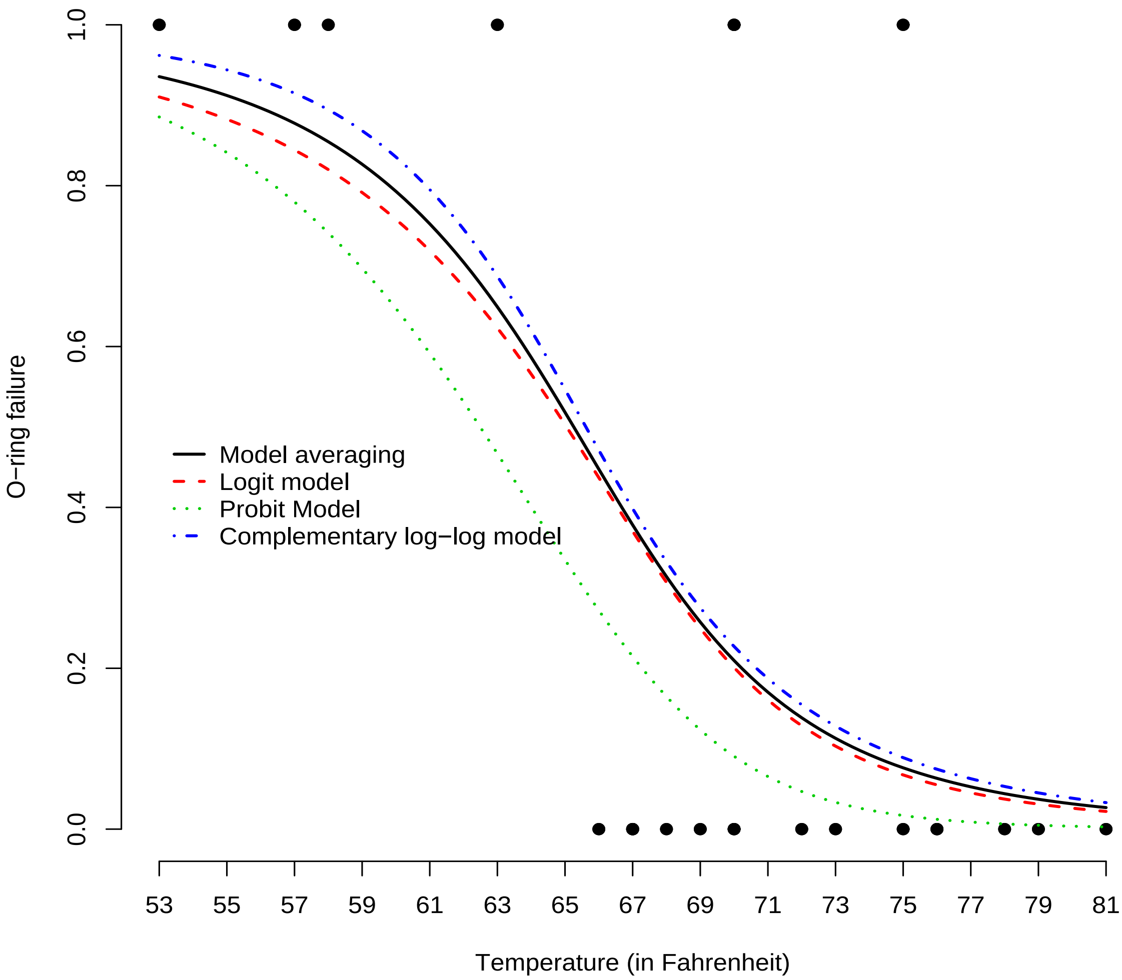
$$
\begin{aligned}
\widehat{Pr}(M_1|y) &= 0.44962664 \\
\widehat{Pr}(M_2|y) &= 0.04160751 \\
\widehat{Pr}(M_3|y) &= 0.50876585
\end{aligned}
$$

showing, again, that the data supports more the logit (M_1) and complementary log-log (M_3) links.

Finally, $Pr(\tilde{y} = 1|\tilde{x})$ can be approximated by

$$\begin{aligned}\widehat{P}r(\tilde{y} = 1|\tilde{x}) &= \widehat{\pi}_1 \sum_{i=1}^{M} Pr(\tilde{y} = 1|\tilde{x}, \beta_{1i}, M_1) \\ &+ \widehat{\pi}_2 \sum_{i=1}^{M} Pr(\tilde{y} = 1|\tilde{x}, \beta_{2i}, M_2) \\ &+ \widehat{\pi}_3 \sum_{i=1}^{M} Pr(\tilde{y} = 1|\tilde{x}, \beta_{3i}, M_3)\end{aligned}$$

where $\beta_{j1}, \ldots, \beta_{jM}$ is a sample from $p(\beta|y, M_j)$ and $j = 1, 2, 3$.



$Pr(\tilde{y} = 1|\tilde{x})$ - Bayesian model averaging (black), logit models (red), probit models (green) and complementary log-log models (blue).