Limited dependent variable models

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Outline

Limited dependent variable models

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When the dependent variable is censored, values in a certain range are all transformed to a single value (Greene, pp.761)

- Household purchases of durable goods (Tobin, 1958)
- The number of extramarital affairs
- The number of hours worked by a woman in the labor force
- The number of arrests after release from prison
- Household expenditure on various commodity groups
- Vacation expenditures

Tobit model

Tobin (1958) Estimation of relationships for limited dependent variables, *Econometrica*, 26, 24-36.

Only y_i is observed.

Censoring rule (truncation):

$$y_i = \begin{cases} y_i^* & y_i^* > 0 \\ 0 & y_i^* \leq 0 \end{cases}$$

Model structure:

$$y_i^* \sim F(y_i^*|x_i, \theta) \quad \text{with p.d.f.} \quad f(y_i^*|x_i, \theta)$$

Likelihood functions

The likelihood function of θ is

$$L(\theta; y, x) = \left\{ \prod_{i:y_i>0} f(y_i|x_i,\theta) \right\} \{F(0|x_i,\theta)\}^k$$

where k is the number of observations where $y_i = 0$.

The complete likelihood is

$$\begin{aligned}
p(y_i|x_i,\theta) &= \int p(y_i, y_i^*|x_i,\theta) dy_i^* \\
&= \int p(y_i|x_i,\theta,y_i^*) f(y_i^*|x_i,\theta) dy_i^* \\
&= p(y_i|x_i,\theta,y_i^* > 0)(1 - F(0|x_i,\theta)) \\
&+ p(y_i|x_i,\theta,y_i^* \leq 0) F(0|x_i,\theta)
\end{aligned}$$

Full conditional distributions

Given y^*, sampling θ is, in general, straightforward:

$$\pi(\theta|y^*, x) \propto \pi(\theta) \prod_{i=1}^{n} f(y_i^*|x_i, \theta)$$

The data augmentation argument leads to:

$$(y_i^*|x_i, y_i, \theta) \sim y_i 1(y_i > 0) + \left[\frac{f(y_i^*|x_i, \theta)}{F(0|x_i, \theta)} 1(y_i^* \leq 0)\right] 1(y_i = 0)$$

In other words, $y_i^* = y_i$ if $y_i > 0$ and

$$(y_i^*|x_i, y_i, \theta) \sim \frac{f(y_i^*|x_i, \theta)}{F(0|x_i, \theta)} \qquad y_i^* \leq 0$$

if $y_i = 0$.

Linear tobit model

In this case,

$$y_i^* \sim N(x_i'\beta, \sigma^2)$$

Prior distribution: $\pi(\beta, \sigma^2) = \pi(\beta)\pi(\sigma^2)$

$$\begin{aligned} \beta &\sim N(\beta_0, V_0) \\ \sigma^2 &\sim IG(\nu_0/2, \nu_0 s_0^2/2) \end{aligned}$$

Full conditional distributions:

$$\begin{aligned} \beta &\sim N(V_\beta(V_0^{-1}\beta_0 + X'y^*), V_\beta) \\ \sigma^2 &\sim IG\left(\frac{\nu_0 + n}{2}, \frac{\nu_0 s_0^2 + (y^* - X\beta)'(y^* - X\beta)}{2}\right) \\ y_i^* &\sim y_i 1(y_i > 0) + N_{[-\infty,0]}(x_i'\beta, \sigma^2)1(y_i = 0) \end{aligned}$$

where $V_\beta^{-1} = V_0^{-1} + \sigma^{-2}(X'X)^{-1}$.

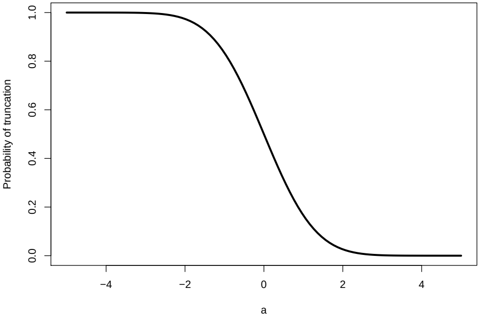
Example

Let $y_i^*|x_i \sim N(\beta x_i, \sigma^2)$ and $x_i \sim N(a, 1)$, such that

$$y_i^* \sim N(\beta a, \sigma^2 + \beta^2)$$

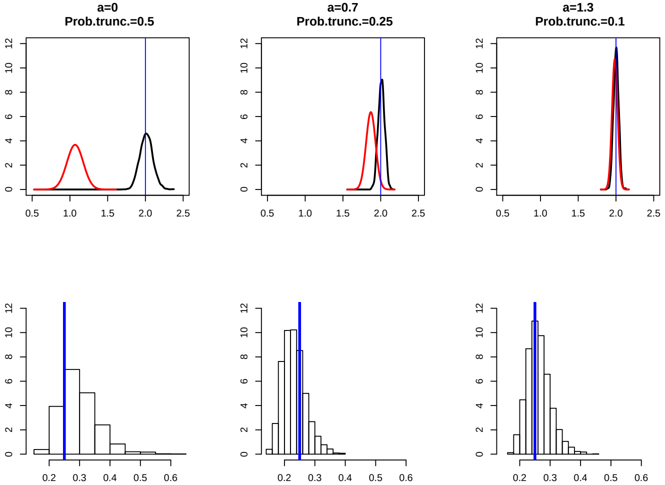
and

$$Pr(y = 0|a, \beta = 2, \sigma^2 = 0.25) = \Phi(-2a/2.062)$$



$\pi(\beta|y,x)$ and $\pi(\sigma^2|y,x)$

$\nu_0 = 0$ and $V_0^{-1} = 0$ (non-informative prior)



(without truncation) and (with truncation)

Probit/ordered probit models

In the ordered probit models, $y_i \in \{1, 2, \dots, J\}$ and

$$y_i = \begin{cases} 1 & \gamma_0 < y_i^* \leq \gamma_1 \\ \vdots & \vdots \\ J & \gamma_{J-1} < y_i^* \leq \gamma_J \end{cases}$$

where $y_i^* \sim F(x_i, \theta)$ with p.d.f. $f(y_i^*|x_i, \theta)$.

Note: In probit models, $y_i \in \{1, 2\}$ and $\gamma_1 = 0$.

Full conditional distributions

Assuming that $\pi(\theta)$ and $\pi(\gamma)$ are uniform in $-\infty = \gamma_0 \leq \cdots \leq \gamma_J = \infty$, then

$$
\begin{aligned}
\pi(\theta|y^*, X) &\propto \pi(\theta) \prod_{i=1}^{n} f(y_i^*|x_i, \theta) \\
(y_i^*|x_i, y_i, \theta) &\sim \sum_{j=1}^{J} \frac{f(y_i^*|x_i, \theta) 1(y_i = j)}{F(\gamma_j|x_i, \theta) - F(\gamma_{j-1}|x_i, \theta)} \\
(\gamma_j|\gamma_{(-j)}, \theta, y^*) &\sim U(\bar{\gamma}_{j-1}, \bar{\gamma}_{j+1})
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_{(-j)} &= (\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_J) \\
\bar{\gamma}_{j-1} &= \max\{\max\{y_i^* : y_i = j\}, \gamma_{j-1}\} \\
\bar{\gamma}_{j+1} &= \min\{\min\{y_i^* : y_i = j+1\}, \gamma_{j+1}\}
\end{aligned}
$$

Multinomial probit models

In multinomial probit models[1], U_{ij} is the utility of individual i when she chooses alternative j, and

$$y_{ij}^* = U_{ij} - U_{i0} = x_{ij}'\beta_j + \varepsilon_{ij}$$

while the econometrician only observes y_i such that

$$y_i = \begin{cases} 0 & \max(y_i^*) < 0 \\ 1 & y_{i1}^* = \max(y_i^*) \geq 0 \\ \vdots & \vdots \\ J-1 & y_{i,J-1}^* = \max(y_i^*) \geq 0 \\ J & y_{i,J}^* = \max(y_i^*) \geq 0 \end{cases}$$

where $y_i \in \{0, 1, \ldots, J\}$ and $y_i^* = (y_{i1}^*, \ldots, y_{iJ}^*)$.

[1] See all reference in the end.

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Stacking up the equations

$$\begin{pmatrix} y_{i1}^* \\ y_{i2}^* \\ \vdots \\ y_{iJ}^* \end{pmatrix} = \begin{pmatrix} x_{i1}' & 0 & \cdots & 0 \\ 0 & x_{i2}' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{iJ}' \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_J \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{iJ} \end{pmatrix}$$

or,

$$y_i^* = X_i\beta + \varepsilon_i$$

with $\varepsilon_i \sim N(0, \Omega)$.

Also,

$$y^* = X\beta + \varepsilon$$

where $y^* = (y_1^*, \ldots, y_N^*)$, $X = (X_1', \ldots, X_N')$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$.

Prior and full conditionals

Prior distribution

$$\begin{aligned} \pi(\beta, \Omega) &= \pi(\beta)\pi(\Omega) \\ \beta &\sim \text{Normal} \\ \Omega &\sim \text{Inverse Wishart} \end{aligned}$$

Full conditional distributions

$$\begin{aligned} \beta &\sim \text{Normal} \\ \Omega &\sim \text{Inverse Wishart} \\ y_i^* &\sim \begin{cases} N(X_i\beta, \Omega)1(\max(y_i^*) < 0) & y_i = 0 \\ N(X_i\beta, \Omega)1(\max(y_i^*) = y_{ij}^* \geq 0) & y_i = j \end{cases} \end{aligned}$$

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